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KAC-RICE FORMULAS FOR RANDOM FIELDS AND THEIR APPLICATIONS IN: RANDOM GEOMETRY, ROOTS OF RANDOM POLYNOMIALS AND SOME ENGINEERING PROBLEMS

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XXX ESCUELA VENEZOLANA DE MATEMÁTICAS

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Kac-Rice formulas for random fields and their applications in: random geometry, roots of random polynomials and some engineering problems.

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To Enrique Cabaña

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Preface

At the end of the seventies of last century in Latin America an intense phenomenon of intellectual migration took place. The dictatorships that were established in the south cone of the continent developed a policy of persecution of professors and researchers of universities and research centers of several countries of this zone. Venezuela, a stable democracy at that time, received many of these researchers.

To this group belong two Uruguayan professors very attached to these notes. We refer to Enrique Cabaña and Mario Wschebor. In Venezuela both researchers addresses the study of the extension, to dimensions greater than one, of the Rice's formula (recently it has been called the Kac-Rice formula to emphasize the parallel discovery of this formula by Marc Kac). At the same time and more or less with the same motivations Robert Adler developed a similar study, generalizing also the mentioned formula.

Three key works appeared in the early eighties. The article of E. Cabaña and that the reader will have the opportunity to revisit with these notes, bears the title "*Esperanzas de integrales sobre conjuntos de nivel aleatorios* [11]", a lecture notes written by M. Wschebor "*Surfaces aléatoires.* LNM 1147 [29]" and finally the book of Robert Adler "*The Geometry of Random Fields*" [1].

Perhaps these themes, "exotic" at the time they were studied, experienced a real revival in the 21st century. These well-established formulas at that time have been re-demonstrated again. However, the most important aspect of this revival is the application of the Kac-Rice formulas in several fields of both pure and applied mathematics. With these notes we want to introduce the participants of the XXX Escuela Venezolana de Matemáticas to an old and at the same time young area.

The authors
Caracas, Montevideo and Grenoble, July 2017

Chapter 1

Preliminaries

1.1 Introduction

There exist two variants of the change of variables formula for multiple integrals very useful in integral geometry. The first one corresponds to smooth, locally bijective functions $G : \mathbb{R}^d \to \mathbb{R}^d$ and the second applies to smooth functions $G : \mathbb{R}^d \to \mathbb{R}^j$ with $d > j$, having a differential with maximal rank. These formulas are called *area formula* and *coarea formula* respectively. Applying these formulas to trajectories of random fields and taking expectation afterwards, one obtains the well-known Kac-Rice formulas. In recent times and fundamentally due to the appearance of two excellent books [2] and [6], there has been a growing interest in the application of these formulas in such varied domains as: random algebraic geometry, algorithm complexity for solving large systems of equations, study of zeros of random polynomial systems and finally, engineering applications. The present work is divided in three parts.

1. In the first part, we give an analytical proof of the area and coarea formulas. Such a proof, originally attributed to Banach and Federer [13], will be made by using elementary tools of vector calculus and measure theory in $\mathbb{R}^d$.

2. The above formulas form the basis for establishing the validity of Kac-Rice formulas for random fields. They allow computing the

expectation of the measure of the level sets

$$\mathcal{C}_{Q,X}(\mathbf{y}) = \{t \in Q \subset \mathbb{R}^d \,:\, X(t) = \mathbf{y}\},$$

where $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ is a random fields and $d \geqslant j$. We must point out that one can obtain a Kac-Rice formula for almost sure all level by using the area and coarea formulas, Fubini theorem and duality. However, in applications the interest is directed to a fixed level $\mathbf{y}$. For instance, the zeros in the study of the roots of a random polynomial. This precision leads us to a delicate study for generalizing the classical inverse function and implicit function theorems. For this part we based our approach in two seminal works: firstly an article of E. Cabaña [11], published in the conference proceedings of the II CLAPEM and secondly in the Lecture Notes of Mathematics of M. Wshebor [29]. The method we use also makes it possible to obtain the Kac-Rice formula for the upper moments of the level measurement.

3. The work ends with several applications. First, we show examples where the hypothesis can be checked and then we use the Kac-Rice formulas for obtaining conditions about the finiteness of the first and second moment of the measure of the level sets. The very important case of the Gaussian random fields leads us to explicitly computations. Afterwards, we address the study of the number of roots of algebraic and trigonometric random polynomials. We emphasize the asymptotic behavior of the expectation and the variance of the number of roots [4]. Particular attention is devoted to systems of random polynomials of several variables that are invariant under the action of the group of rotations in $\mathbb{R}^d$. Another theme we consider is the nodal curves of the system of random waves considered by Berry and Dennis in [9]. These curves are called dislocations in physics and correspond to lines of darkness in light propagation, or threads of silence in sound propagation (*cf.*[9]) . We also study the application of the Kac-Rice formula to sea modeling and to random gravitational lenses.

1.2 Hypothesis and notations

Let D be an open set of $\mathbb{R}^d$. Also let $j \leqslant d$ be a positive integer and $G : D \subset \mathbb{R}^d \to \mathbb{R}^j$ be a function.
The function G satisfies the hypothesis **H1** if:

H1: G is continuously differentiable on D.

We denote $\nabla G(\cdot)$ its Jacobian.
For $\mathbf{y} \in \mathbb{R}^j$ we define the level set at $\mathbf{y}$ as:

$$\mathcal{C}_G(\mathbf{y}) = \{\mathbf{x} \in D : G(\mathbf{x}) = \mathbf{y}\} = G^{-1}(\mathbf{y}),$$

and

$$\mathcal{C}_{Q,G}(\mathbf{y}) = \mathcal{C}_G(\mathbf{y}) \cap Q,$$

where Q is a subset of $\mathbb{R}^d$.
If G satisfies **H1**, we will denote $\mathrm{D}_\mathrm{G}^\mathrm{r}$ the following set

$$\mathrm{D}_\mathrm{G}^\mathrm{r} = \{\mathbf{x} \in D : \nabla G(\mathbf{x}) \text{ is of rank } \ j\}.$$

Also $\mathcal{C}_G^{D^r}(\mathbf{y})$ (resp. $\mathcal{C}_{Q,G}^{D^r}(\mathbf{y})$) denotes the level set, $\mathcal{C}_G^{D^r}(\mathbf{y}) = \mathcal{C}_G(\mathbf{y}) \cap \mathrm{D}_\mathrm{G}^\mathrm{r}$ (resp. $\mathcal{C}_{Q,G}^{D^r}(\mathbf{y}) = \mathcal{C}_{Q,G}(\mathbf{y}) \cap \mathrm{D}_\mathrm{G}^\mathrm{r}$).
From now on, σ_d denotes the Lebesgue measure on $\mathbb{R}^d$. We use the symbol T for the transposition operator. For a set $A \subset \mathbb{R}^d$, A^c will denote its complement on $\mathbb{R}^d$ and if $A \subset D$, A^{c_1} will denote its complement on D. The class of set $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra in $\mathbb{R}^d$. Also $\overline{\mathbb{R}}^+$ is the set of positive real numbers including $+\infty$, $||\cdot||_d$ denotes the Euclidean norm in $\mathbb{R}^d$.
For $\mathbf{x} \in \mathbb{R}^d$, $B(\mathbf{x}, r)$ (resp. $\overline{B}(\mathbf{x}, r)$), $r > 0$, is the open ball (resp. closed) of center $\mathbf{x}$ and radius r, that is $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^d, ||z - x||_d < r\}$ (resp. $\overline{B}(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^d, ||z - x||_d \leqslant r\}$).
$\mathbb{N}^\star = \{x \in \mathbb{Z}, x > 0\}$.
An application $f : (E, d_E) \to (F, d_F)$ between two metric spaces is said to be L-Lipschitz, $L \geqslant 0$, if $d_F(f(\mathbf{x}), f(\mathbf{y})) \leqslant L\, d_E(\mathbf{x}, \mathbf{y})$, for every pair of points $\mathbf{x}, \mathbf{y} \in E$.
We also say that an application is Lipschitz if it is L-Lipschitz for some L.

In the same manner an application $f : (E, d_E) \to (F, d_F)$ between two metric spaces, is said to be locally Lipschitz, if for each $\mathbf{x} \in E$, there exists a neighborhood $V_{\mathbf{x}}$ of $\mathbf{x}$ such that the restriction of function f to $V_{\mathbf{x}}$ is Lipschitz.

$\mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ denotes the vector space of linear functions from $\mathbb{R}^d$ to $\mathbb{R}^j$ with the norm $||\cdot||_{j,d}$. Also $\mathfrak{L}^2(\mathbb{R}^d, \mathbb{R}^j)$ is the vector space of symmetric linear applications continues from $\mathbb{R}^d$ to $\mathbb{R}^j$ with the norm $||\cdot||_{j,d}^{(s)}$. If B is a matrix, B_{ij} denotes the element of i-row and j-column.
For $j \in \mathbb{N}^\star$, S^{j-1} is the boundary of the unit ball of $\mathbb{R}^j$.
For any function f, $\text{supp}(f)$ will be denoted its support.
C will be a generic constant that could change of value in the interior of a proof.

Chapter 2

A proof of the Coarea formula

2.1 Coarea formula

The two results below are known in the literature as the coarea-formula, c.f. Federer [13] pp. 247-249 and Cabaña [11].
Our proof is based on the excellent notes of Weizäcker & Geiβler from Kaiserslautern University [28].

Theorem 2.1.1 *Let $f : \mathbb{R}^j \to \mathbb{R}$ be a mesurable function and $G : D \subset \mathbb{R}^d \to \mathbb{R}^j$, $j \leq d$, be a function satisfying the hypothesis* **H1** *where D is an open set. For all borel set B subset of D the following formula holds:*

$$\int_B f(G(\boldsymbol{x}))(\det(\nabla G(\boldsymbol{x})\nabla G(\boldsymbol{x})^T))^{1/2} d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^j} f(\boldsymbol{y})\, \sigma_{d-j}(\mathcal{C}_{B,G}^{D^r}(\boldsymbol{y}))d\boldsymbol{y}, \tag{2.1}$$

provided that one of the two integrals is finite.

Remark 2.1.1 If f is mesurable and positive the equality (2.1) holds true and in this case the integrales can be infinite.

Remark 2.1.2 The additional hypotheses that f is bounded and B is a compact set imply that the left side integral is finite and the formula (2.1) holds true.

Corollary 2.1.1 *Let h be a mesurable function, $h : \mathbb{R}^d \times \mathbb{R}^j \to \mathbb{R}$ and $G : D \subset \mathbb{R}^d \to \mathbb{R}^j$, $j \leqslant d$, be a function satisfying the hypothesis* **H1** *where D is an open set. For all borel set B subset of D, we have*

$$\int_B h(\boldsymbol{x}, G(\boldsymbol{x}))(\det(\nabla G(\boldsymbol{x})\nabla G(\boldsymbol{x})^T))^{1/2} d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^j} \left[\int_{\mathcal{C}_{B,G}^{D^r}(\boldsymbol{y})} h(\boldsymbol{x}, \boldsymbol{y}) d\sigma_{d-j}(\boldsymbol{x}) \right] d\boldsymbol{y}, \tag{2.2}$$

provided that one of the two integrals is finite.

Remark 2.1.3 If h mesurable and positive the equality (2.2) is satisfied and in this case the integrales can be infinite.

Remark 2.1.4 The hypotheses that h is bounded and B is compact imply that the left side integral of (2.2) is finite and the formula holds.

Proof of Theorem 2.1.1 and Corollary 2.1.1. In first place we will show, along the lines of [28] pp 60-67, the following proposition:

Proposition 2.1.1 *Let $g : \mathbb{R}^d \to \mathbb{R}^j$, $j \leq d$ be a continuously differentiable function defined on $\mathbb{R}^d$. Then for all $A \in \mathcal{B}(\mathbb{R}^d)$ we have*

$$\int_A (\det(\nabla g(\boldsymbol{x})\nabla g(\boldsymbol{x})^T))^{1/2} d\boldsymbol{x} = \int_{\mathbb{R}^j} \sigma_{d-j}(\mathcal{C}_{A,g}(\boldsymbol{y})) d\boldsymbol{y}.$$

Proof of Proposition 2.1.1. As we have observed the proof is based in the notes [28] pp 60-67. In first place we shall prove the formula for affine functions g, then we will consider the formula for sets A of null Lebesgue measure in $\mathbb{R}^d$, afterwards we will consider sets A that are subsets of D_g^r and finally sets including the critical points of g.
To complete our task we will prove before some lemmas.

Lemma 2.1.1 *Proposition 2.1.1 holds true for surjective affine functions g, i.e. if $g(x) = a + \varphi(x)$ where $a \in \mathbb{R}^j$ is fixed and φ is a linear function with maximal rank j.*

Proof of Lemma 2.1.1. Without lose of generality we can always consider $a = 0$. Indeed, on one hand $\nabla g(\cdot) = \nabla \varphi(\cdot)$, thus for all borel set A of

$\mathbb{R}^d$ the following equality holds true:

$$\int_A (\det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T))^{1/2} d\mathbf{x} = \int_A (\det(\nabla \varphi(\mathbf{x})\nabla \varphi(\mathbf{x})^T))^{1/2} d\mathbf{x}.$$

On the other hand, because the mesure σ_j is translation invariant, we have:

$$\int_{\mathbb{R}^j} \sigma_{d-j}(\mathcal{C}_{A,g}(\mathbf{y}))d\mathbf{y}$$

$$= \int_{\mathbb{R}^j} \sigma_{d-j}(\mathcal{C}_{A,\varphi}(\mathbf{y}\text{-}\mathbf{a}))d\sigma_j(\mathbf{y}) = \int_{\mathbb{R}^j} \sigma_{d-j}(\mathcal{C}_{A,\varphi}(\mathbf{y}))d\mathbf{y}.$$

Let now V be the vectorial subspace of $\mathbb{R}^d$ defined by $V = \ker \varphi$. This space has dimension $(d-j)$ because φ is of maximal rank j. We denote as $V^\perp$ its orthogonal that have dimension j. We will work with a coordinates systems associated to these spaces, that is if $\mathbf{x} \in \mathbb{R}^d$, we will write $\mathbf{x} = (\mathbf{z}, \mathbf{w})$ with $\mathbf{z} \in V^\perp$ and $\mathbf{w} \in V$. Then the Lebesgue measure σ_d over $\mathbb{R}^d$ is the product measure $\sigma_j \otimes \sigma_{d-j}$.
Observe that $\varphi/_{V^\perp}$ is a one to one function since $\dim V^\perp = j$. Let us denote Ψ the inverse function of this restriction, that is $\Psi = (\varphi/_{V^\perp})^{-1}$, we have $\varphi \circ \Psi = Id_{\mathbb{R}^j}$ and $\Psi \circ \varphi/V^\perp = Id_{V^\perp}$. Moreover, given that φ^T sends $\mathbb{R}^j$ into $V^\perp$ by definition of V, we have:

$$\Psi^T \circ \Psi \circ \varphi \circ \varphi^T = \Psi^T \circ \varphi^T = (\varphi \circ \Psi)^T = Id_{\mathbb{R}^j},$$

then:

$$\det(\varphi\varphi^T)^{1/2} = \det(\Psi^T\Psi)^{-1/2} = |\det(\Psi)|^{-1}, \tag{2.3}$$

the last equality is a consequence of the fact that Ψ is an endomorphism of $\mathbb{R}^j$.
Let A be a fixed borel set of $\mathbb{R}^d$. Let us consider the function

$$h : V^\perp \longrightarrow \overline{\mathbb{R}}^+$$
$$\mathbf{z} \mapsto \sigma_{d-j}\{\mathbf{w} \in V : (\mathbf{z}, \mathbf{w}) \in A\}.$$

Observe that for $\mathbf{y} \in \mathbb{R}^j$ we have

$$\varphi^{-1}(\mathbf{y}) \cap A = \{(\Psi(\mathbf{y}), \mathbf{w}) : \mathbf{w} \in V\} \cap A$$

and given that $\Psi(\mathbf{y}) \in V^{\perp}$,

$$h(\Psi(\mathbf{y})) = \sigma_{d-j}(\varphi^{-1}(\mathbf{y}) \cap A). \tag{2.4}$$

So, given that the Lebesgue measure σ_d is the product measure $\sigma_j \otimes \sigma_{d-j}$, we get

$$\sigma_d(A) = \int_{V^{\perp}} h(\mathbf{z})\, d\sigma_j(\mathbf{z}). \tag{2.5}$$

Finally as the function φ is a linear function and by using the equalities (2.3), (2.5), the formula of change of variable for function Ψ which is a C^1 function as well as Ψ^{-1} as endomorphism in finite dimension and the equality (2.4), we obtain

$$\begin{aligned}
\int_A (\det(\nabla\varphi(\mathbf{x})\nabla\varphi(\mathbf{x})^T))^{1/2} d\mathbf{x} &= \int_A (\det(\varphi\varphi^T))^{1/2} d\mathbf{x} \\
&= \sigma_d(A)\, |\det(\Psi)|^{-1} \\
&= \int_{V^{\perp}} |\det(\Psi)|^{-1}\, h(\mathbf{z})\, d\sigma_j(\mathbf{z}) \\
&= \int_{\mathbb{R}^j} h(\Psi(\mathbf{y}))\, d\mathbf{y} \\
&= \int_{\mathbb{R}^j} \sigma_{d-j}(\varphi^{-1}(\mathbf{y}) \cap A)\, d\mathbf{y},
\end{aligned}$$

this ends the proof of Lemma 2.1.1 for the affine functions. □

Lemma 2.1.2 *Let $A \subseteq \mathbb{R}^d$ be a borel set and $g : A \longrightarrow \mathbb{R}^j$, $j \leq d$ be a Lipschitz function of Lipschitz constant, $Lip(g)$ where $\mathbb{R}^d$ and $\mathbb{R}^j$ are respectively the d and j dimensional euclidean spaces. Thus for all integer k such that $j \leqslant k \leqslant d$ we have*

$$\int_{\mathbb{R}^j} \sigma_{k-j}(g^{-1}(\boldsymbol{y}) \cap A)\, d\boldsymbol{y} \leqslant \frac{\omega_j \omega_{k-j}}{\omega_k} Lip^j(g)\, \sigma_k(A).$$

Above ω_d denotes the volume of the unit ball of $\mathbb{R}^d$.

Remark 2.1.5 In particular, for $k = d$, we get that Proposition 2.1.1 holds true for the borel sets A of zero Lebesgue measure in $\mathbb{R}^d$.

Proof of the Remark 2.1.5. Given that g is C^1 over $\mathbb{R}^d$ then it is locally Lipschitz over $\mathbb{R}^d$. This last space can be decomposed into a numerable union of disjoint rectangles such that g is Lipschitz over each of them. Thus if A is a borel set of $\mathbb{R}^d$ such that $\sigma_d(A) = 0$, it can be decomposed in an union of disjoint borel sets, of zero Lebesgue measure, such that g est Lipschitz over each of them.
The Lemma 2.1.2 can be applied for $k = d$ for each of these borel sets, this implies that Proposition 2.1.1 is satisfied for all borel set belonging to $\mathbb{R}^d$ and of zero Lebesgue measure over which g is Lipschitz.
Finally given that the measure σ_{d-j} is σ-additive and using the Beppo Levi theorem we have that Remark 2.1.5 holds. □

Proof of Lemma 2.1.2. One can assume that A is bounded. Indeed, it is enough to decompose A as a numerable union of bounded disjoint borel sets and to show the lemma for these borel sets. Given that the measures σ_{d-j} and σ_k are σ-additives, the lemma will be true for A any borel set.
If $\delta > 0$, we will denote by H_k^δ the Hausdorff euclidian pre-measure which defines the Hausdorff euclidian measure of dimension k, denoted by H_k, $k \in \mathbb{N}^\star$. The measure H_k coincides with the Lebesgue measure σ_k on $\mathbb{R}^k$ with the euclidean norm (c.f. [28] page 16).
Set $\delta = \frac{1}{\ell}$, $\ell \in \mathbb{N}^\star$. By definition of $H_k^{\frac{1}{\ell}}$, there exists a covering $((U_i^\ell)_{i\in I_\ell})_{\ell\in\mathbb{N}}$ of A formed by closed sets such that for all ℓ

$$|U_i^\ell| \leqslant \frac{1}{\ell} \quad \text{and} \quad \frac{\omega_k}{2^k} \sum_{i\in I_\ell} |U_i^\ell|^k < H_k^{\frac{1}{\ell}}(A) + \frac{1}{\ell}, \tag{2.6}$$

where $|U|$ denotes the euclidean diameter U, that is

$$|U| = \sup_{x,y\in U} ||x-y||_k.$$

Moreover, by definition of $H_{k-j}^{\frac{1}{\ell}}$, and given that $((U_i^\ell)_{i\in I_\ell})_{\ell\in\mathbb{N}}$ covers A, we have

$$\frac{2^{k-j}}{\omega_{k-j}} H_{k-j}^{\frac{1}{\ell}}(g^{-1}(\mathbf{y}) \cap A) \leqslant \sum_{i\in I_\ell} |U_i^\ell|^{k-j} \mathbb{1}_{g(U_i^\ell)}(\mathbf{y}) = h_\ell(\mathbf{y}). \tag{2.7}$$

Then from the inequality (2.7), the Fatou lemma and the fact that the measures σ_{k-j} and H_{k-j} coincide, we get the following inequalities

$$\begin{aligned}
&\frac{2^{k-j}}{\omega_{k-j}}\int_{\mathbb{R}^j}\sigma_{k-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y} = \frac{2^{k-j}}{\omega_{k-j}}\int_{\mathbb{R}^j}H_{k-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y}\\
&= \frac{2^{k-j}}{\omega_{k-j}}\int_{\mathbb{R}^j}\lim_{\ell\to+\infty}H^{\ell}_{k-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y} \leqslant \int_{\mathbb{R}^j}\liminf_{\ell\to+\infty}h_\ell(\mathbf{y})\,d\mathbf{y}\\
&\leqslant \liminf_{\ell\to+\infty}\int_{\mathbb{R}^j}h_\ell(\mathbf{y})\,d\mathbf{y} = \liminf_{\ell\to+\infty}\int_{\mathbb{R}^j}\sum_{i\in I_\ell}|U_i^\ell|^{k-j}\mathbb{1}_{g(U_i^\ell)}(\mathbf{y})\,d\mathbf{y}\\
&= \liminf_{\ell\to+\infty}\sum_{i\in I_\ell}|U_i|^{\ell^{k-j}}\sigma_j(g(U_i^\ell)). \qquad (2.8)
\end{aligned}$$

The idea is now to establish a relation between $\sigma_j(g(U_i^\ell))$ and $|U_i^\ell|$. The isodiametric inequality for the norms (c.f. [28] page 14) will allow us to get this relation and thus to continue our proof. Let us recall this inequality.

Proposition 2.1.2 *Let C be a borel set of $\mathbb{R}^j$ then*

$$\sigma_j(C) \leqslant \frac{\omega_j}{2^j}|C|^j.$$

Function g is Lipschitz and then continuous over A that is bounded, so that the images $g(U_i^\ell)$ are compact sets hence are bounded.
By using the isodiametric inequality for these bounded sets and using also that g is Lipschitz with Lipschitz constant $Lip(g)$, and to finish inequality (2.6), inequality (2.8) gives us

$$\begin{aligned}
&\frac{2^{k-j}}{\omega_{k-j}}\int_{\mathbb{R}^j}\sigma_{k-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y} \leqslant \liminf_{\ell\to+\infty}\sum_{i\in I_\ell}|U_i^\ell|^{k-j}\sigma_j(g(U_i^\ell))\\
&\leqslant \liminf_{\ell\to+\infty}\sum_{i\in I_\ell}|U_i^\ell|^{k-j}\frac{\omega_j}{2^j}|g(U_i^\ell)|^j \leqslant \liminf_{\ell\to+\infty}\sum_{i\in I_\ell}|U_i^\ell|^{k-j}\frac{\omega_j}{2^j}Lip^j(g)|U_i^\ell|^j\\
&= \liminf_{\ell\to+\infty}\sum_{i\in I_\ell}|U_i^\ell|^k\frac{\omega_j}{2^j}Lip^j(g)\\
&\leqslant \liminf_{\ell\to+\infty}\frac{\omega_j}{2^j}Lip^j(g)\,\frac{2^k}{\omega_k}\left(H_k^{\frac{1}{\ell}}(A)+\frac{1}{\ell}\right)\\
&= 2^{k-j}\frac{\omega_j}{\omega_k}Lip^j(g)\,H_k(A) = 2^{k-j}\frac{\omega_j}{\omega_k}Lip^j(g)\,\sigma_k(A)
\end{aligned}$$

This ends the proof of lemma 2.1.2. □

Lemma 2.1.3 *Proposition 2.1.1 holds if $A \subseteq D_g^r$.*

Proof of Lemma 2.1.3.

We can always assume that A is a compact. Indeed, given that A is a borel set of $\mathbb{R}^d$, then it can be written, except for a zero measure set, as a nondecreasing union of compact.
Remark 2.1.5 following Lemma 2.1.2 and the Beppo Levi theorem allow us to show the Proposition 2.1.1 only in the compact case.

Let us choose an element $x \in A$. Consider the $\mathbb{R}^d$ vectorial subspace defined by $V = \ker \nabla g(x)$. It has $(d-j)$ dimension since $\nabla g(x)$ has maximal rank j. Let $V^{\perp}$ be its orthogonal that is of dimension j. Let us observe that $\nabla g(x)/_{V^{\perp}}$ is one to one.
We will denote by π_V the orthogonal projection of $\mathbb{R}^d$ into V and let define the function $h_x : \mathbb{R}^d \to \mathbb{R}^d$ as

$$h_x(\mathbf{x'}) = x + \pi_V(\mathbf{x'}) + (\nabla g(x)/_{V^{\perp}})^{-1}(g(\mathbf{x'}) - g(x)).$$

We want to prove that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $B_\delta(x)$ is the set

$$B_\delta(x) = \overline{B}(x,\delta) \cap A, \tag{2.9}$$

then

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^d(1+\varepsilon)^j\left(\int_{B_\delta(x)}(\det(\nabla g(\mathbf{x'})\nabla g(\mathbf{x'})^T))^{1/2}d\mathbf{x'} - \varepsilon\,\sigma_d(B_\delta(x))\right)$$

$$\leq \int_{\mathbb{R}^j}\sigma_{d-j}(C_{B_\delta(x),g}(\mathbf{y}))\,d\mathbf{y} \leq \tag{2.10}$$

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^d(1-\varepsilon)^j\left(\int_{B_\delta(x)}(\det(\nabla g(\mathbf{x'})\nabla g(\mathbf{x'})^T))^{1/2}d\mathbf{x'} + \varepsilon\sigma_d(B_\delta(x))\right).$$

For doing so, let us begin by showing the two following things:
For all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\mathbf{x'}, \mathbf{x''} \in B_\delta(x)$, we have:

$$(1-\varepsilon)||\mathbf{x'} - \mathbf{x''}||_d \leq ||h_x(\mathbf{x'}) - h_x(\mathbf{x''})||_d \leq (1+\varepsilon)||\mathbf{x'} - \mathbf{x''}||_d, \tag{2.11}$$

as well as

$$|(\det(\nabla g(x)\nabla g(x)^T))^{1/2} - (\det(\nabla g(\mathbf{x'})\nabla g(\mathbf{x'})^T))^{1/2}| < \varepsilon. \qquad (2.12)$$

The inequality (2.12) is a consequence of the fact that $\nabla g(\cdot)$ is a continuous function defined on A.
To prove the inequality (2.11), let us begin by pointing out that

$$(\nabla g(x)/_{V^\perp})^{-1},$$

is an endomorphism in finite dimension, hence continuous. Thus, we have $||(\nabla g(x)/_{V^\perp})^{-1}||_{j,j} < +\infty$.
Moreover let us define $\Delta_\delta(x)$ as

$$\Delta_\delta(x) = \sup_{\mathbf{x'},\mathbf{x''} \in B_\delta(x)} \frac{||g(\mathbf{x'}) - g(\mathbf{x''}) - \nabla g(x)(\mathbf{x'} - \mathbf{x''})||_j}{||\mathbf{x'} - \mathbf{x''}||_d},$$

recalling that $B_\delta(x)$ is defined by equality (2.9).
Given that g belongs to C^1, we can write the following first order Taylor development

$$g(\mathbf{x'}) = g(\mathbf{x''}) + \left(\int_0^1 \nabla g(\mathbf{x''} + \lambda(\mathbf{x'} - \mathbf{x''}))\, d\lambda\right)(\mathbf{x'} - \mathbf{x''}),$$

getting

$$\begin{aligned} & g(\mathbf{x'}) - g(\mathbf{x''}) - \nabla g(x)(\mathbf{x'} - \mathbf{x''}) \\ & \quad = \left(\int_0^1 (\nabla g(\mathbf{x''} + \lambda(\mathbf{x'} - \mathbf{x''})) - \nabla g(x))\, d\lambda\right)(\mathbf{x'} - \mathbf{x''}), \end{aligned}$$

this implies, since $\nabla g(\cdot)$ is continuous, that for all $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\Delta_\delta(x) \leqslant \varepsilon\, ||(\nabla g(x)/_{V^\perp})^{-1}||_{j,j}^{-1}. \qquad (2.13)$$

Let $\varepsilon > 0$ be a fixed real number and $\mathbf{x'}, \mathbf{x''} \in B_\delta(x)$. Given that $V = \ker \nabla g(x)$, we have

$$\nabla g(x)/_{V^\perp}(\pi_{V^\perp}(\mathbf{x'} - \mathbf{x''})) = \nabla g(x)(\mathbf{x'} - \mathbf{x''}),$$

this implies

$$
\begin{aligned}
&||\mathbf{x'} - \mathbf{x"} - (h_x(\mathbf{x'}) - h_x(\mathbf{x"}))||_d \\
&\quad = \quad ||\pi_{V^\perp}(\mathbf{x'} - \mathbf{x"}) - (\nabla g(x)/_{V^\perp})^{-1}(g(\mathbf{x'}) - g(\mathbf{x"}))||_d \\
&\quad = \quad ||(\nabla g(x)/_{V^\perp})^{-1} \left(\nabla g(x)(\mathbf{x'} - \mathbf{x"}) - (g(\mathbf{x'}) - g(\mathbf{x"})\right)||_d \\
&\quad \leqslant \quad ||(\nabla g(x)/_{V^\perp})^{-1}||_{j,j}\, \Delta_\delta(x)||\mathbf{x'} - \mathbf{x"}||_d \leqslant \varepsilon\, ||\mathbf{x'} - \mathbf{x"}||_d,
\end{aligned}
$$

the last inequality comes from of the inequality (2.13) and ends the proof of (2.11).
Let $T_x : \mathbb{R}^d \to \mathbb{R}^j$ be the following affine function,

$$T_x(\mathbf{x'}) = g(x) + \nabla g(x)(\mathbf{x'} - x).$$

It is surjective because $\nabla g(x)$ is of maximal rank j.
Moreover it holds that $T_x \circ h_x = g$.
Indeed, given that $\pi_V(\mathbf{x'}) \in V$ and $(\nabla g(x)/_{V^\perp})^{-1}(g(\mathbf{x'}) - g(x)) \in V^\perp$, we can write

$$
\begin{aligned}
T_x(h_x(\mathbf{x'})) &= g(x) + \nabla g(x)(h_x(\mathbf{x'}) - x) \\
&= g(x) + \nabla g(x)(\pi_V(\mathbf{x'})) \\
&\qquad + \nabla g(x) \left((\nabla g(x)/_{V^\perp})^{-1}(g(\mathbf{x'}) - g(x))\right) \\
&= g(x) + \nabla g(x)/_{V^\perp} \left((\nabla g(x)/_{V^\perp})^{-1}(g(\mathbf{x'}) - g(x))\right) \\
&= g(x) + g(\mathbf{x'}) - g(x) \\
&= g(\mathbf{x'})
\end{aligned}
$$

Furthermore, the inequality (2.11) allows us to conclude that for fixed $\varepsilon > 0$, h_x is Lipschitz on $B_\delta(x)$, having a Lipschitz constant equal to $(1 + \varepsilon)$. As h_x is an injective function on $B_\delta(x)$, h_x admits an inverse function defined on $h_x(B_\delta(x))$. The inequality (2.11) assures that this inverse is also Lipschitz on $h_x(B_\delta(x))$, an has a Lipschitz constant equal to $(1 - \varepsilon)^{-1}$.
These two facts allow us to apply to h_x and h_x^{-1} the Lipschitz contraction principle that we recall below (c.f. [28] page 18).

Proposition 2.1.3 *Let E, F be two borel subsets of two metric spaces. We assume that there exists a surjective Lipschitz function $f : E \to F$ with Lipschitz constant L. Then*

$$\sigma_k(F) \leqslant L^k\, \sigma_k(E) \ \textit{for all}\ \ k \geqslant 0.$$

Let us apply simultaneously this last principle firstly to the function $f = h_x$ for $E = B_\delta(x)$, $F = h_x(B_\delta(x))$, $L = (1+\varepsilon)$ and $k = d$, and secondly to $f = h_x^{-1}$ for $E = h_x(B_\delta(x))$, $F = B_\delta(x)$, $L = (1-\varepsilon)^{-1}$ and $k = d$, we obtain

$$(1-\varepsilon)^d \, \sigma_d(B_\delta(x)) \leqslant \sigma_d(h_x(B_\delta(x))) \leqslant (1+\varepsilon)^d \, \sigma_d(B_\delta(x)). \qquad (2.14)$$

Let G be a borel set such that $G \subseteq h_x(B_\delta(x))$, let apply again simultaneously the contraction principle to the function $f = h_x^{-1}$, for $E = G$, $F = h_x^{-1}(G)$, $L = (1-\varepsilon)^{-1}$ and $k = d - j$, then to $f = h_x$, for $E = h_x^{-1}(G)$, $F = G$, $L = (1+\varepsilon)$ and $k = d - j$, we get

$$(1+\varepsilon)^{-(d-j)} \, \sigma_{d-j}(G) \leqslant \sigma_{d-j}(h_x^{-1}(G)) \leqslant (1-\varepsilon)^{-(d-j)} \, \sigma_{d-j}(G).$$

In particular if we choose $G = T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x))$, and observing that $T_x \circ h_x = g$, $\sigma_{d-j}(h_x^{-1}(T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x)))) = \sigma_{d-j}(g^{-1}(\mathbf{y}) \cap B_\delta(x))$, we obtain

$$(1+\varepsilon)^{-(d-j)} \, \sigma_{d-j}(T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x))) \leqslant \sigma_{d-j}(g^{-1}(\mathbf{y}) \cap B_\delta(x)) \leqslant$$
$$(1-\varepsilon)^{-(d-j)} \, \sigma_{d-j}(T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x))) \qquad (2.15)$$

Now we can prove the inequality (2.10).
For doing so we apply the Lemma 2.1.1 to the surjective affine function T_x for the borel set $A = h_x(B_\delta(x))$, the inequality (2.15), also the inequalities (2.14) and (2.12), thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y}) \cap B_\delta(x)) \, d\mathbf{y} \\
&\leqslant (1-\varepsilon)^{-(d-j)} \int_{\mathbb{R}^j} \sigma_{d-j}(T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x))) \, d\mathbf{y} \\
&= (1-\varepsilon)^{-(d-j)} \int_{h_x(B_\delta(x))} (\det(\nabla T_x(\mathbf{x}') \nabla T_x(\mathbf{x}')^T))^{1/2} d\mathbf{x}' \\
&= (1-\varepsilon)^{-(d-j)} \, \sigma_d(h_x(B_\delta(x)) \, (\det \nabla g(x) \nabla g(x)^T)^{1/2} \\
&\leqslant (1-\varepsilon)^{-(d-j)} \, (1+\varepsilon)^d \, \sigma_d(B_\delta(x)) (\det \nabla g(x) \nabla g(x)^T)^{1/2} \\
&\leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^d (1-\varepsilon)^j \left(\int_{B_\delta(x)} (\det(\nabla g(\mathbf{x}') \nabla g(\mathbf{x}')^T))^{1/2} d\mathbf{x}' + \varepsilon \sigma_d(B_\delta(x))\right)
\end{aligned}$$

In the same fashion

$$\begin{aligned}
&\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y}) \cap B_\delta(x))\, d\mathbf{y} \\
&\quad \geqslant (1+\varepsilon)^{-(d-j)} \int_{\mathbb{R}^j} \sigma_{d-j}(T_x^{-1}(\mathbf{y}) \cap h_x(B_\delta(x)))\, d\mathbf{y} \\
&\quad = (1+\varepsilon)^{-(d-j)} \int_{h_x(B_\delta(x))} (\det(\nabla T_x(\mathbf{x'}) \nabla T_x(\mathbf{x'})^T))^{1/2} d\mathbf{x'} \\
&\quad = (1+\varepsilon)^{-(d-j)}\, \sigma_d(h_x(B_\delta(x)))\, (\det \nabla g(x) \nabla g(x)^T)^{1/2} \\
&\quad \geqslant (1+\varepsilon)^{-(d-j)}\, (1-\varepsilon)^d\, \sigma_d(B_\delta(x))(\det \nabla g(x) \nabla g(x)^T)^{1/2} \\
&\quad \geqslant \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^d (1+\varepsilon)^j \Big(\int_{B_\delta(x)} (\det(\nabla g(\mathbf{x'}) \nabla g(\mathbf{x'})^T))^{1/2} d\mathbf{x'} \\
&\qquad\qquad -\varepsilon\, \sigma_d(B_\delta(x))\Big).
\end{aligned}$$

To end the proof of Lemma 2.1.3, for fixed $\varepsilon > 0$, using the Vitali covering theorem (c.f. [28], Theorem 1. 15 page 14), the set A can be covered, except for a set of zero Lebesgue measure, by a sequence of disjoint sets of the type $B_\delta(x)$. These sets are closed because A is closed being a compact. Since by Lemma 2.1.2 we can forget the zero measure set. Then by taking the sum over this partition we obtain by using the inequality (2.10) and the compactness of A that implies $\sigma_d(A) < +\infty$

$$\begin{aligned}
&\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^d (1+\varepsilon)^j \left(\int_A (\det(\nabla g(\mathbf{x'}) \nabla g(\mathbf{x'})^T))^{1/2} d\mathbf{x'} - \varepsilon\, \sigma_d(A)\right) \\
\leq\ & \int_{\mathbb{R}^j} \sigma_{d-j}(\mathcal{C}_{A,g}(\mathbf{y}))\, d\mathbf{y} \\
\leq\ & \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^d (1-\varepsilon)^j \left(\int_A (\det(\nabla g(\mathbf{x'}) \nabla g(\mathbf{x'})^T))^{1/2} d\mathbf{x'} + \varepsilon\, \sigma_d(A)\right).
\end{aligned}$$

Given that $\varepsilon > 0$ is small enough, Proposition 2.1.1 is satisfied if $A \subseteq D_g^r$, ending the proof of Lemma 2.1.3. □

Let us consider now the critical points. This leads us to prove the following lemma.

Lemma 2.1.4 *Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a set such that $A \subset (D_g^r)^c$ then $\sigma_{d-j}(A \cap g^{-1}(\boldsymbol{y})) = 0$ for almost surely $\boldsymbol{y} \in \mathbb{R}^j$.*

Proof of Lemma 2.1.4. The idea is perturbing a little the function g in such a way that it becomes of maximal rank but still having a small

value of $(\det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T))^{1/2}$.
We begin increasing the dimension of $\mathbb{R}^d$.
In the same form that in the proof of the Lemma 2.1.3 we can assume that A is a compact set and then the partial derivatives $\frac{\partial g_i}{\partial x_k}$ are uniformly continuous on A.
Let $\varepsilon > 0$ be a fixed real. We extend the function $g : \mathbb{R}^d \to \mathbb{R}^j$ to $\widetilde{g} : \mathbb{R}^{d+j} \to \mathbb{R}^j$, seting $\widetilde{g}(\mathbf{x},\mathbf{z}) = g(\mathbf{x}) + \varepsilon\,\mathbf{z}$.
In this form $\nabla\widetilde{g}(\mathbf{x},\mathbf{z})$ is given by the matrix of dimension $j \times (d+j)$

$$(\nabla g(\mathbf{x}) \quad \varepsilon I_j), \tag{2.16}$$

where I_j is the identity matrix of order j and thus

$$\begin{aligned}\nabla\widetilde{g}(\mathbf{x},\mathbf{z})\nabla\widetilde{g}(\mathbf{x},\mathbf{z})^T &= (\nabla g(\mathbf{x})\varepsilon I_j)\begin{pmatrix}\nabla g(\mathbf{x})^T \\ \varepsilon I_j\end{pmatrix} \\ &= \nabla g(\mathbf{x})\nabla g(\mathbf{x})^T + \varepsilon^2 I_j. \end{aligned} \tag{2.17}$$

Let us see that for $(\mathbf{x},\mathbf{z}) \in A \times \mathbb{R}^j$,

$$0 < (\det(\nabla\widetilde{g}(\mathbf{x},\mathbf{z})\nabla\widetilde{g}(\mathbf{x},\mathbf{z}))^T))^{1/2} \leqslant \mathbf{C}\,\varepsilon, \tag{2.18}$$

where $\mathbf{C}$ is a constant depending on the uniform bound of the bounded partial derivatives $\frac{\partial g_i}{\partial x_k}$ on A.
Firstly we have

$$\Big(\det(\nabla\widetilde{g}(\mathbf{x},\mathbf{z})\nabla\widetilde{g}(\mathbf{x},\mathbf{z}))^T) > 0\Big) \iff (\text{ rang } (\nabla\widetilde{g}(\mathbf{x},\mathbf{z})) = j)$$

and by equality (2.16) since the rank of εI_j is j, the same holds for the rank of $\nabla\widetilde{g}(\mathbf{x},\mathbf{z})$.
Now using equality (2.17), and denoting by X_i the i-th column of $\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T$ and if $(e_i)_{1\leqslant i\leqslant j}$ is the canonical basis of $\mathbb{R}^j$, we can write using $\det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T) = 0$, that

$$\begin{aligned}&\det(\nabla\widetilde{g}(\mathbf{x},\mathbf{z})\nabla\widetilde{g}(\mathbf{x},\mathbf{z})^T) \\ &\qquad= \det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T + \varepsilon^2 I_j) \\ &\qquad= \det(X_1 + \varepsilon^2 e_1, X_2 + \varepsilon^2 e_2, \ldots, X_j + \varepsilon^2 e_j) \\ &\qquad= \det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T) + \varepsilon^2\, B_\varepsilon(X_1, X_2, \ldots, X_j) \\ &\qquad= \varepsilon^2\, B_\varepsilon(X_1, X_2, \ldots, X_j),\end{aligned}$$

where $B_\varepsilon(X_1, X_2, \ldots, X_j)$ is uniformly bounded on A since it is a compact set.
Then we have proved the inequality (2.18).
Since $\widetilde{g}$ belongs to C^1 on $\mathbb{R}^{d+j}$ and that it is of maximal rank on $A \times \mathbb{R}^j$, we can apply the Lemma 2.1.3.
But before, we have to compare the fibers of g and those of $\widetilde{g}$.
Let Q be the unity ball of $\mathbb{R}^j$, then for all $\mathbf{y} \in \mathbb{R}^j$, we have

$$\begin{aligned}\widetilde{g}^{-1}(\mathbf{y}) \cap (A \times Q) &= \{(\mathbf{x}, \mathbf{z}) \in A \times Q : g(\mathbf{x}) = \mathbf{y} - \varepsilon \mathbf{z}\} \\ &= \bigcup_{\mathbf{z} \in Q} g^{-1}((\mathbf{y} - \varepsilon\, \mathbf{z}) \cap A) \times \{\mathbf{z}\}\end{aligned}$$

Thus for all pair of points $\mathbf{y}$, $\mathbf{z}$ of $\mathbb{R}^j$, it holds

$$\begin{aligned}&\sigma_{d-j}(g^{-1}(\mathbf{y} - \varepsilon\, \mathbf{z}) \cap A)\, \mathbb{1}_Q(\mathbf{z}) \\ &\qquad = \sigma_{d-j}(\pi^{-1}(\mathbf{z}) \cap \widetilde{g}^{-1}(\mathbf{y}) \cap (A \times Q)),\end{aligned} \tag{2.19}$$

where $\pi : \mathbb{R}^{d+j} \to \mathbb{R}^j$ is the canonical projection.
Now let us apply the coarea formula to $\widetilde{g}$, that is Lemma 2.1.3, for the borel set $A \times Q$, thus

$$\begin{aligned}\int_{R^j} \sigma_d(\widetilde{g}^{-1}(\mathbf{y}) \cap (A \times Q))\, d\mathbf{y} &= \int_{A \times Q} (\det(\nabla \widetilde{g}(\mathbf{x}, \mathbf{z}) \nabla \widetilde{g}(\mathbf{x}, \mathbf{z})^T))^{1/2}\, d\mathbf{x}\, d\mathbf{z} \\ &\leqslant \mathbf{C}\, \varepsilon\, \sigma_d(A),\end{aligned} \tag{2.20}$$

the last bound comes from the inequality (2.18) and from the fact that $\sigma_j(Q) = 1$.
The idea is now to come back to the function g and to apply the equality (2.19). To make this we will apply the Lemma 2.1.2 for a fixed $\mathbf{y}$ and to the projection π, which is a contracting function thus a fortiori a Lipschitz function with Lipschitz constant $L = 1$. The borel set of $\mathbb{R}^{d+j}$ to which we apply the lemma is $\widetilde{g}^{-1}(\mathbf{y}) \cap (A \times Q)$ for the real $k = d$ that satisfies $j \leqslant k \leqslant d + j$. We obtain for fixed $\mathbf{y}$ and using the equality (2.19)

$$\int_Q \sigma_{d-j}(g^{-1}(\mathbf{y}-\varepsilon\,\mathbf{z})\cap A)\,d\mathbf{z}$$

$$\begin{aligned} &= \int_{\mathbb{R}^j} \sigma_{d-j}(\pi^{-1}(\mathbf{z})\cap \widetilde{g}^{-1}(\mathbf{y})\cap (A\times Q))\,d\mathbf{z} \\ &\leqslant \omega_j\,\frac{\omega_{d-j}}{\omega_d}\,\sigma_d(\widetilde{g}^{-1}(\mathbf{y})\cap (A\times Q)). \end{aligned}$$

By integrating this last inequality with respect to $\mathbf{y}$ on $\mathbb{R}^j$ and using the inequality (2.20), we have

$$\int_{\mathbb{R}^j}\int_Q \sigma_{d-j}(g^{-1}(\mathbf{y}-\varepsilon\,\mathbf{z})\cap A)\,d\mathbf{z}\,d\mathbf{y} \leqslant \omega_j\,\frac{\omega_{d-j}}{\omega_d}\,\mathbf{C}\,\varepsilon\,\sigma_d(A).$$

Applying the Fubini-Tonelli theorem and the fact that $\sigma_j(Q) = 1$, it holds

$$\int_{\mathbb{R}^j}\int_Q \sigma_{d-j}(g^{-1}(\mathbf{y}-\varepsilon\,\mathbf{z})\cap A)\,d\mathbf{z}\,d\mathbf{y}$$

$$\begin{aligned} &= \int_Q\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y}-\varepsilon\,\mathbf{z})\cap A)\,d\mathbf{y}\,d\mathbf{z} \\ &= \int_Q\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y}\,d\mathbf{z} = \int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y}. \end{aligned}$$

Thus we have shown

$$\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y} \leqslant \omega_j\,\frac{\omega_{d-j}}{\omega_d}\,\mathbf{C}\,\varepsilon\,\sigma_d(A).$$

Given that $\varepsilon > 0$ is small enough and $\sigma_d(A) < +\infty$ because A is compact, we finally have proved that

$$\int_{\mathbb{R}^j} \sigma_{d-j}(g^{-1}(\mathbf{y})\cap A)\,d\mathbf{y} = 0.$$

This implies that $\sigma_{d-j}(g^{-1}(\mathbf{y})\cap A) = 0$, for almost surely $\mathbf{y}\in\mathbb{R}^j$.
This ends the proof of Lemma 2.1.4. □

To finish the proof of Proposition 2.1.1, it remains to decompose $A \in \mathcal{B}(\mathbb{R}^d)$, under the form $A = \left(A\cap D_g^r\right)\cup\left(A\cap (D_g^r)^c\right)$, noting that D_g^r is a borel set of $\mathbb{R}^d$ because $(D_g^r)^c = \{x\in\mathbb{R}^d, \det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T) = 0\}$, that is the inverse image of $\{0\}$ for the continue function

$$\det(\nabla g(\cdot)\nabla g(\cdot)^T).$$

We just apply Lemma 2.1.3 to the borel set $\left(A \cap D_g^r\right)$ and Lemma 2.1.4 to the borel set $\left(A \cap (D_g^r)^c\right)$.
Finishing the proof of Proposition 2.1.1. □
We can prove now the Theorem 2.1.1 and its corollary.
We begin by proving the following theorem.

Theorem 2.1.2 *Let $f : \mathbb{R}^j \to \mathbb{R}$ be a positive measurable function and $g : \mathbb{R}^d \to \mathbb{R}^j$, $j \leq d$, be a continuously differentiable function. For all borel set Q of $\mathbb{R}^d$ the following formula holds true:*

$$\int_Q f(g(\boldsymbol{x}))(\det(\nabla g(\boldsymbol{x})\nabla g(\boldsymbol{x})^T))^{1/2} d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^j} f(\boldsymbol{y})\, \sigma_{d-j}(\mathcal{C}_{Q,g}^{D^r}(\boldsymbol{y}))d\boldsymbol{y}, \tag{2.21}$$

the integrals could be eventually infinite.

Remark 2.1.6 Theorem 2.1.2 remains true if one assumes that the function g is only locally Lipschitz on $\mathbb{R}^d$ instead of being C^1 on $\mathbb{R}^d$. In this case, the function g is almost surely differentiable on $\mathbb{R}^d$ and the measure σ_{d-j} that appears into the right-side of the equality (2.21) is replaced by the euclidian Hausdorff H_{d-j} measure. We invite the reader to consult the Theorem 4.12 page 61 of [28] for more details.

Proof of Theorem 2.1.2. Set $A = \mathrm{D}_\mathrm{g}^\mathrm{r} \cap g^{-1}(I) \cap Q$ where I is a borel set of $\mathbb{R}^j$ and Q is a borel set of $\mathbb{R}^d$. Given that A is a borel set of $\mathbb{R}^d$ the Proposition 2.1.1 allows us to have the following formula:

$$\int_Q \mathbf{1}_I(g(\mathbf{x}))(\det(\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T))^{1/2} d\mathbf{x} = \int_{\mathbb{R}^j} \mathbb{1}_I(\mathbf{y})\, \sigma_{d-j}(\mathcal{C}_{Q,g}^{D^r}(\mathbf{y}))\, d\mathbf{y},$$

and the equality (2.21) holds true for the corresponding functions, this is for the functions f of the form $f = \mathbb{1}_I$ and also for measurable step functions positives.
The Beppo Levi theorem assures that this equality holds true for positive measurable functions. Yielding Theorem 2.1.2. □
Let us prove now the Theorem 2.1.1 and the Corollary 2.1.1.
This needs the following two lemmas.

Lemma 2.1.5 *Let $G : D \subset \mathbb{R}^d \to \mathbb{R}^j$, be a locally Lipschitz function defined on D an open set of $\mathbb{R}^d$. Then the function G is Lipschitz over all compact set K that is contained in D.*

Proof of the Lemma 2.1.5. Let us consider $\mathbf{x} \in K \subset D$. Since G is locally Lipschitz on the open set D, there exists a constant $L_{\mathbf{x}} > 0$ and a radius $r_{\mathbf{x}} > 0$, such that $B(\mathbf{x}, r_{\mathbf{x}}/2) \subset B(\mathbf{x}, r_{\mathbf{x}}) \subset D$ and satisfying, for all $\mathbf{u}, \mathbf{v} \in B(\mathbf{x}, r_{\mathbf{x}})$, $||G(\mathbf{u}) - G(\mathbf{v})||_j \leqslant L_{\mathbf{x}}||\mathbf{u} - \mathbf{v}||_d$.
Since G is locally Lipschitz on D, it is continuous on D and also on the compact set K. Set $M = \sup_{\mathbf{u}\in K} ||G(\mathbf{u})||_j < +\infty$.
Since K is compact, there exists $m \in \mathbb{N}^*$, such that for all $i = 1, m$, there exists a $\mathbf{x}_i \in K$, verifying that $K \subset \cup_{i=1}^{m} B(\mathbf{x_i}, r_{\mathbf{x_i}}/2) \subset D$.
Set also $L = \max_{i=1,m} L_{\mathbf{x}_i}$ and $r = \min_{i=1,m} r_{\mathbf{x}_i}$. Let us prove that G is a Lipschitz function on K with Lipschitz constant $\widetilde{L}$ which is least or equal to $max(L, \frac{4M}{r})$.
Indeed, consider $\mathbf{u}, \mathbf{v} \in K$.
If $||\mathbf{u} - \mathbf{v}||_d \leqslant \frac{r}{2}$; there exists $i, j \in 1, \cdots, m$, such that $\mathbf{u} \in B(\mathbf{x_i}, r_{\mathbf{x_i}}/2)$ and $\mathbf{v} \in B(\mathbf{x_j}, r_{\mathbf{x_j}}/2)$. In this case $\mathbf{u}, \mathbf{v} \in B(\mathbf{x_i}, r_{\mathbf{x_i}})$ and

$$||G(\mathbf{u}) - G(\mathbf{v})||_j \leqslant L_{\mathbf{x_i}}||\mathbf{u} - \mathbf{v}||_d \leqslant L||\mathbf{u} - \mathbf{v}||_d.$$

If $||\mathbf{u} - \mathbf{v}||_d > \frac{r}{2}$, $||G(\mathbf{u}) - G(\mathbf{v})||_j \leqslant 2M \leqslant \frac{4M}{r} \times \frac{r}{2} \leqslant \frac{4M}{r}||\mathbf{u} - \mathbf{v}||_d$.
This ends the proof of the lemma. □

Lemma 2.1.6 *Let $G : K \subset \mathbb{R}^d \to \mathbb{R}^j$, be a k-Lipschitz function where K is a compact set of $\mathbb{R}^d$. Then the function G admits an extension $g : \mathbb{R}^d \to \mathbb{R}^j$ that is a $\sqrt{j}k$-Lipschitz function.*

Proof of the Lemma 2.1.6. Let us point out that we can consider the case where the function G takes real values, consider $G : K \subset \mathbb{R}^d \to \mathbb{R}$ and k- Lipschitz.
In fact, if $G = (G_1, G_2, \cdots, G_j) : K \subset \mathbb{R}^d \to \mathbb{R}^j$ is k-Lipschitz, then for all $i = 1, 2, \cdots, j$, the functions $G_i : K \subset \mathbb{R}^d \to \mathbb{R}$ are k-Lipschitz. If we show that these functions can be extended to $\mathbb{R}^d$ by the functions g_i that still are k-Lipschitz then the function $g = (g_1, g_2, \cdots, g_j) : \mathbb{R}^d \to \mathbb{R}^j$ will be an extension of G that is $\sqrt{j}k$-Lipschitz.
Let then $G : K \subset \mathbb{R}^d \to \mathbb{R}$ be a k-Lipschitz function, let us extend it to a

function $g : \mathbb{R}^d \to \mathbb{R}^j$, k-Lipschitz.
For $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in K$, set:

$$G_{\mathbf{y}}(\mathbf{x}) = G(\mathbf{y}) + k||\mathbf{x} - \mathbf{y}||_d \text{ and } \quad g(\mathbf{x}) = \inf_{\mathbf{y} \in K} G_{\mathbf{y}}(\mathbf{x}).$$

Let us remark first that the function g is well defined.
In fact, for all $\mathbf{x} \in \mathbb{R}^d$, $\{G_{\mathbf{y}}(\mathbf{x}), \mathbf{y} \in K\}$ is a subset of $\mathbb{R}$, bounded below by $\inf_{\mathbf{y} \in K} G(\mathbf{y})$. This number exists because the function G is Lipschitz on K then continue on K.
Let us show that the function g extends the function G.
For all $\mathbf{x} \in K$, $g(\mathbf{x}) = \inf_{\mathbf{y} \in K} G_{\mathbf{y}}(\mathbf{x}) \leqslant G_{\mathbf{x}}(\mathbf{x}) = G(\mathbf{x})$.
Moreover, $\forall\, \mathbf{x} \in K$, $\forall\, n \in \mathbb{N}^*$, $\exists\, \mathbf{y_n} \in K$, $G_{\mathbf{y_n}}(\mathbf{x}) \leqslant g(\mathbf{x}) + \frac{1}{n}$, then
$G(\mathbf{y_n}) + k||\mathbf{x} - \mathbf{y_n}||_d \leqslant g(\mathbf{x}) + \frac{1}{n}$.
But G is k-Lipschitz on K then $G(\mathbf{x}) \leqslant G(\mathbf{y_n}) + k||\mathbf{x} - \mathbf{y_n}||_d$.
Hence we have for all $\mathbf{x} \in K$, for all $n \in \mathbb{N}^*$, $G(\mathbf{x}) \leqslant g(\mathbf{x}) + \frac{1}{n}$. Finally one shows that if $\mathbf{x} \in K$, $g(\mathbf{x}) = G(\mathbf{x})$.
The only thing that remains to prove is that the above function g is k-Lipschitz on $\mathbb{R}^d$.
Firstly, let us observe that for all $\mathbf{y} \in K$, the function $G_{\mathbf{y}}$ is k-Lipschitz on $\mathbb{R}^d$. Indeed, for all $\mathbf{x} \in \mathbb{R}^d$, for all $\mathbf{z} \in \mathbb{R}^d$,

$$|G_{\mathbf{y}}(\mathbf{x}) - G_{\mathbf{y}}(\mathbf{z})| = k|||\mathbf{x} - \mathbf{y}||_d - ||\mathbf{z} - \mathbf{y}||_d| \leqslant k||\mathbf{x} - \mathbf{z}||_d.$$

Thus for all $\mathbf{y} \in K$, for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$,

$$g(\mathbf{x}) = \inf_{\mathbf{y} \in K} G_{\mathbf{y}}(\mathbf{x}) \leqslant G_{\mathbf{y}}(\mathbf{x}) \leqslant G_{\mathbf{y}}(\mathbf{z}) + k||\mathbf{x} - \mathbf{z}||_d,$$

in consequence for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$,

$$g(\mathbf{x}) \leqslant \inf_{\mathbf{y} \in K} G_{\mathbf{y}}(\mathbf{z}) + k||\mathbf{x} - \mathbf{z}||_d,$$

that is,

$$g(\mathbf{x}) - g(\mathbf{z}) \leqslant k||\mathbf{x} - \mathbf{z}||_d.$$

Thus by symmetry one obtains that for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$,

$$|g(\mathbf{x}) - g(\mathbf{z})| \leqslant k||\mathbf{x} - \mathbf{z}||_d,$$

that ends the proof. □

Let us come back to the proof of the Theorem 2.1.1 and Corollary 2.1.1. We consider for all $n \in \mathbb{N}^\star$, the closed sets

$$D_n = \{\mathbf{x} \in \mathbb{R}^d,\, d(\mathbf{x}, D^c) \geqslant \frac{1}{n}\}.$$

Since D is open, the sets $(D_n)_{n \in \mathbb{N}^*}$ are included in D and then for all $n \in \mathbb{N}^\star$, the sets K_n, defined as $K_n = D_n \cap [-n, n]^d$ are compact subsets of D.
Since the function G belongs to C^1 on D that is an open set it is locally Lipschitz on D and from the Lemma 2.1.5, for all $n \in \mathbb{N}^*$ it is Lipschitz on the compact K_n.
The lemma 2.1.6 allows extending for $n \in \mathbb{N}^*$, the Lipschitz function G/K_n to the function $g_n : \mathbb{R}^d \to \mathbb{R}^j$ still Lipschitz on $\mathbb{R}^d$.
Let $f : \mathbb{R}^j \to \mathbb{R}$ be a measurable positive function and B a borel set subset of D. Applying for all $n \in \mathbb{N}^*$, the Remark 2.1.6 to the function g_n, which is Lipschitz on $\mathbb{R}^d$ and a fortiori locally Lipschitz on $\mathbb{R}^d$, and to the borel set $Q_n = K_n \cap B$, we have

$$\int_{Q_n} f(g_n(\mathbf{x}))(\det(\nabla g_n(\mathbf{x}) \nabla g_n(\mathbf{x})^T))^{1/2} d\mathbf{x}$$

$$= \int_{\mathbb{R}^j} f(\mathbf{y})\, H_{d-j}(\mathcal{C}^{D^r}_{Q_n, g_n}(\mathbf{y})) d\mathbf{y},$$

and since $g_n = G$ on K_n then on Q_n, we get

$$\int_{Q_n} f(G(\mathbf{x}))(\det(\nabla G(\mathbf{x}) \nabla G(\mathbf{x})^T))^{1/2} d\mathbf{x} = \int_{\mathbb{R}^j} f(\mathbf{y})\, \sigma_{d-j}(\mathcal{C}^{D^r}_{Q_n, G}(\mathbf{y})) d\mathbf{y}.$$

Moreover, when n tends to infinite, the sets $(Q_n)_{n \in \mathbb{N}^*}$ tend increasingly towards B. The Beppo Levi theorem implies the Remark 2.1.1 and also the Theorem 2.1.1.
The Remark 2.1.2 is a consequence of the fact that if B is a compact set and the function f is bounded, then

$$\int_B |f(G(\mathbf{x}))|(\det(\nabla G(\mathbf{x}) \nabla G(\mathbf{x})^T))^{1/2} d\mathbf{x}$$

$$\leqslant \mathbf{C} \int_B (\det(\nabla G(\mathbf{x}) \nabla G(\mathbf{x})^T))^{1/2} d\mathbf{x} < +\infty,$$

since the function G is C^1 on D and B is a compact subset of D.
Remark 2.1.1 allows us to prove Corollary 2.1.1.
Indeed applying this remark to the measurable and positive function $f = \mathbb{1}_A$ and to the borel set $B \cap Q$ where A (resp. B) is some borel set of $\mathbb{R}^j$ (resp. $\mathbb{R}^d$), and Q is a borel set of $\mathbb{R}^d$, allows to establish the Corollary 2.1.1 for the functions of the form $\mathbb{1}_{B\times A}$ and also for positive measurable step functions h , then for positive measurable functions. Yielding Remark 2.1.3 and Corollary 2.1.1.
In the same manner as for Remark 2.1.2 we get Remark 2.1.4.
This ends the proof of the Theorem 2.1.1 and the Corollary 2.1.1. □

2.2 Kac-Rice formulas for a.s. level

In this section $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) denotes a random field that belongs to $\mathbf{C}^1(D, \mathbb{R}^j)$, $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ will be a continuous process and D is an open set of $\mathbb{R}^d$.
Let H be the operator

$$\begin{aligned} H : \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j) &\longrightarrow \mathbb{R}^+ \\ \mathbf{A} &\longmapsto (\det(\mathbf{A}\mathbf{A}^T))^{1/2}. \end{aligned}$$

Let us recall that the random set $\mathrm{D}_X^{\mathrm{r}}$ was defined as $\mathrm{D}_X^{\mathrm{r}} = \{\mathbf{x} \in D : \nabla \mathrm{X}(\mathbf{x}) \text{ have rank } j\}$ and the level set $\mathbf{y} \in \mathbb{R}^j$, $\mathcal{C}_X^{\mathrm{D}^{\mathrm{r}}}(\mathbf{y})$, was defined as $\mathcal{C}_X^{\mathrm{D}^{\mathrm{r}}}(\mathbf{y}) = \mathcal{C}_X(\mathbf{y}) \cap \mathrm{D}_X^{\mathrm{r}}$.

Let us consider the following hypotheses

- $\mathbf{H_1}$: The function

$$\mathbf{u} \longmapsto \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{u})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right],$$

is a continuous function of the variable $\mathbf{u}$.

- $\mathbf{H_1^\star}$: The function

$$\mathbf{u} \longmapsto \mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\mathbf{u}))\right],$$

is a continuous function of the variable $\mathbf{u}$.

- $\mathbf{H_2}$: The function

$$\mathbf{u} \longmapsto \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\mathbb{E}[|Y(\mathbf{x})|H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\, d\mathbf{x},$$

is a continuous function of the variable $\mathbf{u}$.

- $\mathbf{H_2^\star}$: The function

$$\mathbf{u} \longmapsto \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\mathbb{E}[H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\, d\mathbf{x},$$

is a continuous function of the variable $\mathbf{u}$.

- $\mathbf{H_3}$: For almost surely $\mathbf{x} \in D$, the density of $X(\mathbf{x})$, $\mathrm{p}_{X(\mathbf{x})}(\cdot)$ exists.

By using the coarea formula and by duality, we will prove the following proposition.

Proposition 2.2.1 *1. If X satisfies the hypotheses ($\mathbf{H_1^\star}$ or $\mathbf{H_2^\star}$) and $\mathbf{H_3}$, then for almost surely $\boldsymbol{y} \in \mathbb{R}^j$,*

$$\mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\boldsymbol{y}))\right] = \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx. \tag{2.22}$$

2. If X and Y satisfy the hypotheses ($\mathbf{H_1}$ or $\mathbf{H_2}$) and $\mathbf{H_3}$, then for almost surely $\boldsymbol{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]$$

$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x})H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx. \tag{2.23}$$

Proof of the Proposition 2.2.1. Let us start by proving part 1) of the proposition.

Applying the Remark 2.1.1 following the Theorem 2.1.1 to the function $G = X$ and $f = \mathbb{1}_A$ where A is a borel set of $\mathbb{R}^j$ and to the borel set $B = D$, we have

$$\int_D \mathbb{1}_{X(\mathbf{x})\in A}\, H(\nabla X(\mathbf{x}))\, d\mathbf{x} = \int_A \sigma_{d-j}(\mathcal{C}_X^{D^r}(\mathbf{y}))\, d\mathbf{y}.$$

By taking expectation of each side of the equality, that is possible because the two terms are positives, and applying the Beppo Levi theorem, we get using the hypothesis $\mathbf{H_3}$

$$\int_A \mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\mathbf{y}))\right] d\mathbf{y}$$

$$= \int_A \left[\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}\right] d\mathbf{y}.$$

In this step of the proof we need a duality lemma.

Lemma 2.2.1 *Let $f_1, f_2 : \mathbb{R}^j \to \overline{\mathbb{R}}^+$, be two measurable functions such that for all $A \in \mathcal{B}(\mathbb{R}^j)$, A bounded, $\int_A f_1(\mathbf{y})\, d\mathbf{y} = \int_A f_2(\mathbf{y})\, d\mathbf{y} < +\infty$, then $f_1 = f_2$ σ_j-almost surely.*

Proof of the Lemma 2.2.1. We begin by proving the lemma for two measurable functions g_1 et g_2 taking values on $\overline{\mathbb{R}}^+$ such that, for all $B \in \mathcal{B}(\mathbb{R}^j)$, $\int_B g_1(\mathbf{y})\, d\mathbf{y} = \int_B g_2(\mathbf{y})\, d\mathbf{y} < +\infty$. We consider the set $B = \{g_2 < g_1\}$. The hypothesis $\int_B g_1(\mathbf{y})\, d\mathbf{y} = \int_B g_2(\mathbf{y})\, d\mathbf{y}$, implies, since $\int_{\mathbb{R}^j} g_2(\mathbf{y})\, d\mathbf{y} < +\infty$, $\int_{\mathbb{R}^j} \mathbb{1}_B(\mathbf{y})\, (g_1(\mathbf{y}) - g_2(\mathbf{y}))\, d\mathbf{y} = 0$, and thus $\mathbb{1}_B(\mathbf{y})\, (g_1(\mathbf{y}) - g_2(\mathbf{y})) = 0$ for almost surely $\mathbf{y} \in \mathbb{R}^j$. In the same form we consider $B' = \{g_1 < g_2\}$, concluding $\mathbb{1}_{B'}(\mathbf{y})\, (g_2(\mathbf{y}) - g_1(\mathbf{y})) = 0$ for almost surely $\mathbf{y} \in \mathbb{R}^j$ and finally $g_1 = g_2$, σ_j almost surely.
Considering now two functions f_1 and f_2 satisfying the hypotheses of the lemma. Let K be a compact in $\mathbb{R}^j$. Since for all $B \in \mathcal{B}(\mathbb{R}^j)$, $A = B \cap K$ is a bounded borel set of $\mathbb{R}^j$, and if $g_1 = f_1 \mathbb{1}_K$ and $g_2 = f_2 \mathbb{1}_K$, we have by hypothesis $\int_B g_1(\mathbf{y})\, d\mathbf{y} = \int_B g_2(\mathbf{y})\, d\mathbf{y} < +\infty$. The preliminary result that we have shown implies that for all compact set K of $\mathbb{R}^j$, $f_1 \mathbb{1}_K = f_2 \mathbb{1}_K$, σ_j almost surely.
The proof ends remarking that except a zero measure set the set $\mathbb{R}^j$ can be written as a non-decreasing union of compacts and by applying the Beppo Levi theorem. □

We apply here the lemma to the function $f_1(\mathbf{y}) = \mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\mathbf{y}))\right]$ and to the function $f_2(\mathbf{y}) = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] d\mathbf{x}$.
The function f_1 or f_2 is locally integrable by the assumption $\mathbf{H_1^\star}$ or $\mathbf{H_2^\star}$.
This finishes the proof of part 1) of the proposition.

Let us prove the part 2). In a first time we assume that X et Y satisfy the hypotheses $\mathbf{H_1}$ and $\mathbf{H_3}$. Let us apply the Corollary 2.1.1 to the function $G = X$, $h(\mathbf{x}, \mathbf{y}) = \mathbb{1}_A(\mathbf{y}) \times Y(\mathbf{x}) \times \mathbb{1}_D(\mathbf{x})$ where A is a bounded borel set of $\mathbb{R}^j$ and to the borel set $B = D$, we have almost surely

$$\int_D \mathbb{1}_{X(x)\in A}\, Y(\mathbf{x})\, H(\nabla X(\mathbf{x}))\, d\mathbf{x} = \int_A \left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x})\, d\sigma_{d-j}(\mathbf{x})\right] d\mathbf{y}. \quad (2.24)$$

Indeed, the hypothesis $\mathbf{H_1}$ implies

$$\mathbb{E}\left(\int_A \left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})|\, d\sigma_{d-j}(\mathbf{x})\right] d\mathbf{y}\right) < +\infty, \quad (2.25)$$

thus almost surely $\int_A \left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})|\, d\sigma_{d-j}(\mathbf{x})\right] d\mathbf{y} < +\infty$. Let us remark that the equality (2.24) is still true replacing Y for $|Y|$.
This last observation and the equality (2.25) imply

$$\mathbb{E}\left[\int_D \mathbb{1}_{X(x)\in A}\, |Y(\mathbf{x})|\, H(\nabla X(\mathbf{x}))\, d\mathbf{x}\right] < +\infty.$$

We can take the expectation in both side of the equality (2.24) and from the hypothesis $\mathbf{H_3}$, we obtain that for all bounded borel set A of $\mathbb{R}^j$

$$\int_A \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x})\, d\sigma_{d-j}(\mathbf{x})\right] d\mathbf{y} =$$
$$\int_A \left(\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathbf{dx}\right) d\mathbf{y}.$$

Let us remark that the last equality is still true replacing Y by $|Y|$ and the corresponding integrales are finite.
Consider now

$$f_1(\mathbf{y}) = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x})\, d\sigma_{d-j}(\mathbf{x})\right]$$

and

$$f_2(\mathbf{y}) = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathbf{dx}.$$

The functions f_1 and f_2 a priori do not take their values on $\overline{\mathbb{R}}^+$. However, a small modification of Lemma 2.2.1 can be done, remarking that $\int_A |f_1(\mathbf{y})|\, d\mathbf{y} < +\infty$ et $\int_A |f_2(\mathbf{y})|\, d\mathbf{y} < +\infty$, for all bounded borel set A of $\mathbb{R}^j$, this implies $f_1 = f_2$, σ_{d-j} almost surely.
This ends the proof of the Proposition 2.2.1 in the case where X and Y satisfy the hypotheses $\mathbf{H_1}$ and $\mathbf{H_3}$. A similar proof can be made when X and Y satisfy the hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$. $\square$

Chapter 3

Kac-Rice formula for all level

3.1 Rice formula for a level set regular

Let us point out that the formulas (2.22) et (2.23) hold true for almost surely $\mathbf{y} \in \mathbb{R}^j$. However in applications these formulas are needed for all $\mathbf{y}$ fixed in $\mathbb{R}^j$. We are going to establish a theorem which will give hypotheses on X and Y in such a way that these formulas will be valid for all $\mathbf{y}$ in $\mathbb{R}^j$.
More precisely we will assume the continuity of the two members in the equalities (2.22) and (2.23), restricting us to the set $\mathrm{D}_X^{\mathrm{r}}$ and proving the equality for all $\mathbf{y}$ fixed in $\mathbb{R}^j$.
Before continuing let us establish two hypotheses useful for what follows.

- $\mathbf{H_4}$: The function

$$\mathbf{u} \longmapsto \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{u})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right],$$

  is a continuous function of the variable $\mathbf{u}$.

- $\mathbf{H_5}$: The function

$$\mathbf{u} \longmapsto \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{u}) \mathbb{E}[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{u}]\, d\mathbf{x},$$

  is a continuous function of the variable $\mathbf{u}$.

Theorem 3.1.1 *1. If X satisfies the hypotheses $\mathbf{H_1^\star}$, $\mathbf{H_2^\star}$ and $\mathbf{H_3}$, then for all $\boldsymbol{y} \in \mathbb{R}^j$,*

$$\mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\boldsymbol{y}))\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx. \tag{3.1}$$

2. If X and Y satisfy the hypotheses ($\mathbf{H_1}$ or $\mathbf{H_2}$), $\mathbf{H_3}$, $\mathbf{H_4}$ and $\mathbf{H_5}$, then for all $\boldsymbol{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x})H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx. \tag{3.2}$$

Remark 3.1.1 If X and Y satisfy the hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ and $\mathbf{H_3}$, then for all $\mathbf{y} \in \mathbb{R}^j$ it holds,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})|H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Proof of Theorem 3.1.1 and of the Remark 3.1.1. Let us beging proving the formula (3.1).
Since X satisfies the hypotheses $\mathbf{H_1^\star}$, $\mathbf{H_2^\star}$ and $\mathbf{H_3}$, as a consequence of part 1) of the Proposition 2.2.1, we know that for almost surely $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_X^{D^r}(\mathbf{y}))\right] = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

The hypotheses $\mathbf{H_1^\star}$ and $\mathbf{H_2^\star}$ imply the continuity of each member of the equality and in consequence their equality for all $\mathbf{y} \in \mathbb{R}^j$.

Reasoning in a similar way we can prove the formula (3.2). Ending in this form the proof of Theorem 3.1.1.
The Remark 3.1.1 comes from the fact that the hypotheses $\mathbf{H_4}$ and $\mathbf{H_5}$ transform into the hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, replacing Y by $|Y|$. □

3.1.1 Checking the hypotheses

We have proven in the precedent section the equality (3.2) assuming principally the continuity of its two members.
Our goal in what follows is to give a large class of processes X and Y that satisfies the hypotheses $\mathbf{H_i}$, $i = 1, 5$.
We consider in first place the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$. This leads us to prove the Theorem 3.1.1 which is needed for proving Proposition 3.1.1 that follows. We need to point out that the proof that we will give is deeply inspired by Cabaña [11].
For a while the functions X and Y will be assumed deterministic, this is they are not random functions.

Theorem 3.1.2 *Let $X : D \subset \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a function belonging to $\boldsymbol{C}^1(D, \mathbb{R}^j)$ such that ∇X is Lipschitz on D which is an open and convex set of $\mathbb{R}^d$. Let D_1 be an open and bounded subset of D and $Y : D_1 \subset \mathbb{R}^d \to \mathbb{R}$ a continuous function such that* $\operatorname{supp}(Y) \subset D^r_{X/D_1}$. *Then the function*

$$\boldsymbol{y} \longmapsto \int_{\mathcal{C}^{D^r}_{D_1,X}(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})$$

is continuous with respect to the variable $\boldsymbol{y}$.

Proof of Theorem 3.1.2. In the same fashion as Cabaña we will define an atlas of $\mathcal{C}^{D^r}_X(\mathbf{y})$. Consider $\mathbf{x}_0 \in D^r_X$ fixed, such that $\nabla X(\mathbf{x}_0)$ has rank j. If $A_d = \{1, 2, \ldots, d\}$, there exist $\lambda = (\ell_1, \ell_2, \ldots, \ell_j) \in A^j_d$, $\ell_1 < \ell_2 < \cdots < \ell_j$ such that

$$J^{(\lambda)}_X(\mathbf{x}_0) = \det\left(\frac{\partial(X_1, \ldots, X_j)}{\partial(x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_j})}(\mathbf{x}_0)\right) \neq 0.$$

Set λ^c the complementary index in A_d, that is $\lambda^c = (i_1, i_2, \cdots, i_{d-j}) \in A^{d-j}_d$, and $i_1 < i_2 < \cdots < i_{d-j}$. If $(e_1, e_2, \cdots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, we denote $V_\lambda = vect(e_{i_1}, e_{i_2}, \cdots, e_{i_{d-j}})$ and $V^\perp_\lambda$ its orthogonal, this is $V^\perp_\lambda = vect(e_{\ell_1}, e_{\ell_2}, \cdots, e_{\ell_j})$. With these notations, if $\mathbf{x} = (x_1, x_2, \cdots, x_d) = \sum_{i=1}^{d} x_i e_i \in \mathbb{R}^d$, we denote $\widehat{\mathbf{x}}_\lambda = (x_{i_1}, x_{i_2}, \cdots, x_{i_{d-j}})$.
Let us consider the function f_λ defined from $D \subset \mathbb{R}^d$ into $\mathbb{R}^d$ such that

$\mathbf{x} \to f_\lambda(\mathbf{x}) = \pi_{V_\lambda}(\mathbf{x}) + \sum_{k=1}^{j} (X_k(\mathbf{x}) - y_k) e_{\ell_k}$, where X_k (resp. y_k) denotes the components of X (resp. $\mathbf{y}$), $k = 1, \cdots, j$.
The Jacobian $J_{f_\lambda}(\mathbf{x}_0)$ of this transformation evaluated at the point $\mathbf{x}_0$ is $J_{f_\lambda}(\mathbf{x}_0) = |J_X^\lambda(\mathbf{x}_0)| \neq 0$. By the inverse function theorem there exists an open neighborhood $U_{\mathbf{x}_0}^\lambda$ of $\mathbf{x}_0$ included in D, such that $f_\lambda(U_{\mathbf{x}_0}^\lambda)$ is still an open set of $\mathbb{R}^d$ and such that the restriction $f_\lambda|_{U_{\mathbf{x}_0}^\lambda}$ has an inverse h_λ belonging to C^1 defined from $f_\lambda(U_{\mathbf{x}_0}^\lambda)$ onto $U_{\mathbf{x}_0}^\lambda$.
Let define the set $R_{\mathbf{x}_0}^\lambda$, by

$$R_{\mathbf{x}_0}^\lambda = \{(x_{i_1}, x_{i_2}, \cdots, x_{i_{d-j}}) \in \mathbb{R}^{d-j} : \sum_{k=1}^{d-j} x_{i_k} e_{i_k} \in f_\lambda(U_{\mathbf{x}_0}^\lambda)\}.$$

Since $f_\lambda(U_{\mathbf{x}_0}^\lambda)$ is an open set, the set $R_{\mathbf{x}_0}^\lambda$ is also an open set of $\mathbb{R}^{d-j}$. Let denote $h_\lambda = (h_1^\lambda, h_2^\lambda, \cdots, h_d^\lambda)$. We have the following sequence of equivalences

$$\begin{aligned}
\left(\mathbf{x} \in U_{\mathbf{x}_0}^\lambda, X(\mathbf{x}) = \mathbf{y}\right) &\iff \left(\mathbf{x} \in U_{\mathbf{x}_0}^\lambda, f_\lambda(\mathbf{x}) = \pi_{V_\lambda}(\mathbf{x})\right) \iff \\
&\left(\pi_{V_\lambda}(\mathbf{x}) \in f_\lambda(U_{\mathbf{x}_0}^\lambda), \mathbf{x} = h_\lambda(\pi_{V_\lambda}(\mathbf{x}))\right) \iff \\
&\left(\mathbf{x} = \pi_{V_\lambda}(\mathbf{x}) + \sum_{k=1}^{j} h_{\ell_k}^\lambda(\pi_{V_\lambda}(\mathbf{x}))\, e_{\ell_k}, \widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_0}^\lambda\right) \iff \\
&\left(\mathbf{x} = \sum_{k=1}^{d-j} x_{i_k} e_{i_k} + \sum_{k=1}^{j} h_{\ell_k}^\lambda(\sum_{k=1}^{d-j} x_{i_k} e_{i_k}) e_{\ell_k}, \widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_0}^\lambda\right) \iff \\
&\left(\mathbf{x} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda), \widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_0}^\lambda\right),
\end{aligned}$$

where we have defined $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}} : R_{\mathbf{x}_0}^\lambda \subset \mathbb{R}^{d-j} \to \mathbb{R}^d$ by

$$\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(x_{i_1}, x_{i_2}, \cdots, x_{i_{d-j}}) = \sum_{k=1}^{d-j} x_{i_k} e_{i_k} + \sum_{k=1}^{j} h_{\ell_k}^\lambda(\sum_{k=1}^{d-j} x_{i_k} e_{i_k}) e_{\ell_k}. \tag{3.3}$$

This provides to us a local parametrization of the level set $\mathcal{C}_X^{D^r}(\mathbf{y})$, defined by $\overrightarrow{\alpha_\lambda}$. Moreover such a function belongs to C^1 defined over $R_{\mathbf{x}_0}^\lambda$.

Remark 3.1.2 Furthermore as a bonus we get that $\mathcal{C}_X^{D^r}(\mathbf{y})$ is a differentiable manifold of dimension $d - j$.

Let us mention that if $\mathbf{x_0} \in \mathcal{C}_X^{D^r}(\mathbf{y})$ then $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_{0,\lambda}) = \mathbf{x}_0$.
We now decompose D_X^r in the following form: $D_X^r = \cup_{\lambda \in B_j} \Gamma(\lambda)$, where $B_j = \{\lambda = (\ell_1, \ell_2, \ldots, \ell_j),\, \ell_k \in A_d,\, \ell_1 < \ell_2 < \ldots < \ell_j\}$ and $\Gamma(\lambda) = \{\mathbf{x} \in D,\, J_X^{(\lambda)}(\mathbf{x}) \neq 0\}$.

Remark 3.1.3 For all $\lambda \in B_j$,
$\Gamma(\lambda) = \{\mathbf{x} \in D,\, \nabla X(\mathbf{x})/V_\lambda^\perp \text{ is invertible}\}$

Remark 3.1.4 We could have proved a less general result than that established in this theorem. More exactly we would have been able to show this theorem under the following weaker hypothesis: $Y : D_X^r \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that $\operatorname{supp}(Y) \subset \Gamma(\lambda)$, for any $\lambda \in B_j$. Indeed it is this condition we finally need in the proof of the Propositions 3.1.1 and 3.2.1 given farther. However this result seemed to us interesting in itself because we did not find it in the literature. Furthermore this general result presents the advantage that we exhibit in a neat way a partition of the unity of D_X^r. This construction will allow us afterward in the proof of Proposition 3.2.1 to decompose the function Y on this partition and thus to come down in case where the function has its support included in $\Gamma(\lambda)$.

Let us prove the Theorem 3.1.2 in the case where $D = D_1$. That is when D is an open, convex and bounded set of $\mathbb{R}^d$, $X : D \subset \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is a function $\mathbf{C}^1(D, \mathbb{R}^j)$ such that ∇X is Lipschitz and $Y : D \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that $\operatorname{supp}(Y) \subset D_X^r$. It is enough to prove theorem in the case where $Y : D_X^r \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function with $\operatorname{supp}(Y) \subset D_X^r$. Let $\mathbf{y}$ fixed in $\mathbb{R}^j$.
In first place we assume $\operatorname{supp}(Y) \subset \Gamma(\lambda)$, $\lambda \in B_j$. We will define the integral of Y over the level set $\mathcal{C}_X^{D^r}(\mathbf{y})$, that is we shall give a sense to $\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x})\, \mathrm{d}\sigma_{d-j}(\mathbf{x})$.
Consider $\mathbf{x}_0 \in \operatorname{supp}(Y) \subset \Gamma(\lambda)$. For the precedent facts, there exists an open neighborhood $U_{\mathbf{x}_0}^\lambda$ of $\mathbf{x}_0$, such that
$\left(\mathbf{x} \in U_{\mathbf{x}_0}^\lambda \cap \mathcal{C}_X^{D^r}(\mathbf{y})\right) \iff \left(\mathbf{x} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda),\, \widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_0}^\lambda\right)$.
Since $U_{\mathbf{x}_0}^\lambda$ is an open set, one can chose a radius $r_{\mathbf{x}_0}^\lambda > 0$ such that the closed ball of $\mathbb{R}^d$ with center $\mathbf{x}_0$ and radius $r_{\mathbf{x}_0}^\lambda$ is contained in $U_{\mathbf{x}_0}^\lambda$, let $\overline{B}(\mathbf{x}_0, r_{\mathbf{x}_0}^\lambda) \subset U_{\mathbf{x}_0}^\lambda$.
Since $\operatorname{supp}(Y)$ is a compact set of $\mathbb{R}^d$, we can cover $\operatorname{supp}(Y)$ by a finite

number of balls, that is $\text{supp}(Y) \subset \cup_{i=1}^{m} B(\mathbf{x}_i, r_{\mathbf{x}_i}^{\lambda})$ such that for all $i = 1, \cdots, m$, we still have: $\overline{B}(\mathbf{x}_i, r_{\mathbf{x}_i}^{\lambda}) \subset U_{\mathbf{x}_i}^{\lambda}$.
We will construct a partition of unity for $\text{supp}(Y)$ compact manifold and let us denote it as $\{\pi_1, \cdots, \pi_m\}$.
We define as in the book of Wendell Fleming [15] the C^∞ real variable function h by

$$h(x) = \begin{cases} \exp(\dfrac{-1}{1-x^2}), \ |x| < 1\,; \\ 0, \ |x| \geqslant 1. \end{cases}$$

For $i = 1, \cdots, m$ and for $\mathbf{x} \in \text{supp}(Y)$, let set $\Psi_i(\mathbf{x}) = h\left(\frac{||\mathbf{x}-x_i||_d}{r_{x_i}^{\lambda}}\right)$.
Since $\text{supp}(Y) \subset \cup_{i=1}^{m} B(\mathbf{x}_i, r_{\mathbf{x}_i}^{\lambda})$ by construction we have that for all $\mathbf{x} \in \text{supp}(Y)$, $\sum_{i=1}^{m} \Psi_i(\mathbf{x}) > 0$.
Let define now for $i = 1, \cdots, m$ and for $\mathbf{x} \in \text{supp}(Y)$, $\pi_i(\mathbf{x}) = \frac{\Psi_i(\mathbf{x})}{\sum_{i=1}^{m} \Psi_i(\mathbf{x})}$.
The functions $\{\pi_1, \cdots, \pi_m\}$ define a partition of unity for $\text{supp}(Y)$ because we have

1. π_i is C^∞ over $\text{supp}(Y)$, $\pi_i \geqslant 0$, $i = 1, \cdots, m$.
2. $\text{supp}(\pi_i) = \text{supp}(\Psi_i) \subset \text{supp}(Y) \cap \overline{B}(\mathbf{x}_i, r_{\mathbf{x}_i}^{\lambda}) \subset U_{\mathbf{x}_i}^{\lambda}$.
3. $\sum_{i=1}^{m} \pi_i(\mathbf{x}) = 1$, $\mathbf{x} \in \text{supp}(Y)$.

The integral of Y over the level set $\mathbf{y}$ can be defined by

$$\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}) = \sum_{i=1}^{m} \int_{U_{\mathbf{x}_i}^{\lambda} \cap \mathcal{C}_X^{D^r}(\mathbf{y})} \pi_i(\mathbf{x}) Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})$$
$$= \sum_{i=1}^{m} \int_{R_{\mathbf{x}_i}^{\lambda}} \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)) Y(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))$$
$$\times (\det(\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)^T))^{1/2} \, \mathrm{d}\widehat{\mathbf{x}}_\lambda. \tag{3.4}$$

However, if the level changes we need to modify the procedure. We continue following the way of Cabaña.
Let consider $\mathbf{x}_0 \in \Gamma(\lambda) \cap \mathcal{C}_X^{D^r}(\mathbf{y})$ fixed. We build then $U_{\mathbf{x}_0}^{\lambda}$ and $R_{\mathbf{x}_0}^{\lambda}$.
Let define the function

$$G : R_{\mathbf{x}_0}^{\lambda} \times \mathbb{R}^j \times \widetilde{W}_{\mathbf{x}_0}^{\lambda} \subset \mathbb{R}^{d-j} \times \mathbb{R}^j \times \mathbb{R}^j \to \mathbb{R}^j,$$

by

$$G(\widehat{\mathbf{x}}_\lambda, \delta, \gamma) = X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_k e_{\ell_k}) - \mathbf{y} - \delta,$$

where $\gamma = (\gamma_1, \cdots, \gamma_j)$, $\widehat{\mathbf{x}}_\lambda = (x_{i_1}, x_{i_2}, \cdots, x_{i_{d-j}})$ and $\widetilde{W}^\lambda_{\mathbf{x}_0}$ is a neighborhood of zero such that for all $\widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_0}$, $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_k e_{\ell_k} \in D$ and $\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_k e_{\ell_k}) / V_\lambda^\perp$ is invertible, that remains possible since $\Gamma(\lambda)$ is an open set of D and $\mathbf{x}_0 \in \Gamma(\lambda)$.
Since X is C^1 and $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}$ is C^1 over $R^\lambda_{\mathbf{x}_0}$, then G is C^1 over $R^\lambda_{\mathbf{x}_0} \times \widetilde{W}^\lambda_{\mathbf{x}_0} \times \mathbb{R}^j$.
Moreover, we have $G(\widehat{\mathbf{x}}_{0,\lambda}, 0, 0) = X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_{0,\lambda})) - \mathbf{y} = X(\mathbf{x}_0) - \mathbf{y} = \vec{0}_{\mathbb{R}^j}$.
And

$$\frac{\partial G}{\partial \gamma}(\widehat{\mathbf{x}}_\lambda, \delta, \gamma) = \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_k e_{\ell_k}) / V_\lambda^\perp, \tag{3.5}$$

this implies

$$\frac{\partial G}{\partial \gamma}(\widehat{\mathbf{x}}_{0,\lambda}, 0, 0) = \nabla X(\mathbf{x}_0) / V_\lambda^\perp,$$

that is invertible by Remark 3.1.3.
The implicit function theorem can be applied. Thus there exists three neighborhoods, in first place $V^\lambda_{\mathbf{x}_0} \subset R^\lambda_{\mathbf{x}_0}$ that we can select equal to $R^\lambda_{\mathbf{x}_0}$, and two other neighborhoods of zero in $\mathbb{R}^j$ denote as $W^\lambda_{\mathbf{x}_0}$ and $\widetilde{W}^\lambda_{\mathbf{x}_0}$ (we can also choose this neighborhood equal to $\widetilde{W}^\lambda_{\mathbf{x}_0}$) and a function $\gamma_{\lambda,\mathbf{x}_0}$ belonging to C^1 defined on $R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0}$ onto $\widetilde{W}^\lambda_{\mathbf{x}_0}$, that is $\gamma_{\lambda,\mathbf{x}_0} : R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0} \subset \mathbb{R}^{d-j} \times \mathbb{R}^j \to \widetilde{W}^\lambda_{\mathbf{x}_0} \subset \mathbb{R}^j$ such that

1. $\gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_{0,\lambda}, 0) = \vec{0}_{\mathbb{R}^j}$
2. $\forall (\widehat{\mathbf{x}}_\lambda, \delta) \in R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0}$, $G(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)) = \vec{0}_{\mathbb{R}^j}$
3. $\forall (\widehat{\mathbf{x}}_\lambda, \delta, \gamma) \in R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0} \times \widetilde{W}^\lambda_{\mathbf{x}_0}$,
   $\Big(G(\widehat{\mathbf{x}}_\lambda, \delta, \gamma) = \vec{0}_{\mathbb{R}^j} \Rightarrow \gamma = \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)\Big)$

Moreover, derivating the expression $G(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)) = \vec{0}_{\mathbb{R}^j}$ with respect to $\widehat{\mathbf{x}}_\lambda$, we get for all $(\widehat{\mathbf{x}}_\lambda, \delta) \in R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0}$,

$$O_{j,d-j} =$$

$$\frac{\partial G}{\partial \widehat{\mathbf{x}}_\lambda}(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)) + \frac{\partial G}{\partial \gamma}(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)) \times \frac{\partial \gamma_{\lambda,\mathbf{x}_0}}{\partial \widehat{\mathbf{x}}_\lambda}(\widehat{\mathbf{x}}_\lambda, \delta),$$

where $O_{j,d-j}$ is the null matrix with j rows and $d-j$ columns.
From the equality (3.5) and since

$$\frac{\partial G}{\partial \widehat{\mathbf{x}}_\lambda}(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta))$$

$$= \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_{k,\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta) e_{\ell_k}) \times \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda),$$

finally we obtain

$$\frac{\partial \gamma_{\lambda,\mathbf{x}_0}}{\partial \widehat{\mathbf{x}}_\lambda}(\widehat{\mathbf{x}}_\lambda, \delta) = -(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_{k,\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta) e_{\ell_k}) / V_\lambda^\perp)^{-1} \times$$
$$\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_{k,\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta) e_{\ell_k}) \times \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) \quad (3.6)$$

Defining

$$\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda) = \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) + \sum_{k=1}^{j} \gamma_{k,\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta) e_{\ell_k}, \quad (3.7)$$

this function is a local parametrization of the level set $\mathcal{C}_X^{D^r}(\mathbf{y}+\delta)$. With this new parametrization we can write (3.6) under the form

$$\frac{\partial \gamma_{\lambda,\mathbf{x}_0}}{\partial \widehat{\mathbf{x}}_\lambda}(\widehat{\mathbf{x}}_\lambda, \delta)$$
$$= -(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) / V_\lambda^\perp)^{-1} \times \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) \times \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda). \quad (3.8)$$

In a similar manner by derivating the equality $G(\widehat{\mathbf{x}}_\lambda, \delta, \gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, \delta)) = \vec{0}_{\mathbb{R}^j}$ with respect to δ, we get this time

$$\frac{\partial \gamma_{\lambda,\mathbf{x}_0}}{\partial \delta}(\widehat{\mathbf{x}}_\lambda, \delta) = (\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) / V_\lambda^\perp)^{-1} \quad (3.9)$$

We need to point out that if $\widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_0}$ then $G(\widehat{\mathbf{x}}_\lambda, 0, 0) = X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)) - \mathbf{y} = \vec{0}_{\mathbb{R}^j}$. By the precedent point 3. we obtain that $\gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, 0) = 0$, and since $\gamma_{\lambda,\mathbf{x}_0}$ is a continuous function on $R^\lambda_{\mathbf{x}_0} \times W^\lambda_{\mathbf{x}_0}$, we get firstly

$$\lim_{\delta \to 0} \overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda) = \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda).$$

Being the convergence uniform over $\overline{R}^\lambda_{\mathbf{x}_0}$. In fact since $\gamma_{\lambda,\mathbf{x}_0}(\widehat{\mathbf{x}}_\lambda, 0) = 0$ and by using the mean value theorem and the equality (3.9), we have

$$\begin{aligned}
&||\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda) - \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)||_d \\
&\quad = \; ||\sum_{k=1}^{j}(\sum_{i=1}^{j} \frac{\partial \gamma_{k,\lambda,\mathbf{x}_0}}{\partial \delta_i}(\widehat{\mathbf{x}}_\lambda, \theta_k \delta)\delta_i) e_{\ell_k}||_d \\
&\quad = \; ||\sum_{k=1}^{j}((\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\theta_k\delta}}(\widehat{\mathbf{x}}_\lambda))/V^\perp_\lambda)^{-1}(\delta))_{k,1}\, e_{\ell_k}||_d \\
&\quad \leqslant \; \sqrt{j} \sup_{\mathbf{z} \in K} ||(\nabla X(\mathbf{z})/V^\perp_\lambda)^{-1}||_{d,j} \times ||\delta||_j,
\end{aligned}$$

where $0 < \theta_k < 1$, for $k = 1, \cdots, j$ and K a compact set of $\mathbb{R}^d$ that is defined by $K = \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\overline{R}^\lambda_{\mathbf{x}_0}) + \sum_{i=1}^{j} \gamma_{\ell_{k,\lambda,\mathbf{x}_0}}(\overline{R}^\lambda_{\mathbf{x}_0} \times \overline{W}^\lambda_{\mathbf{x}_0}) e_{\ell_k}$. To finish, let us recall that $||(\nabla X(\mathbf{z})/V^\perp_\lambda)^{-1}||$ remains bounded over this compact. To show this it is enough to make smaller the open sets $V^\lambda_{\mathbf{x}_0}$ and $W^\lambda_{\mathbf{x}_0}$. That is chosing $V^\lambda_{\mathbf{x}_0}$ such that $\overline{V}^\lambda_{\mathbf{x}_0} \subset R^\lambda_{\mathbf{x}_0}$ and chosing an open set $F^\lambda_{\mathbf{x}_0}$ containing 0 on $\mathbb{R}^j$ such that $\overline{F}^\lambda_{\mathbf{x}_0} \subset W^\lambda_{\mathbf{x}_0}$.
Secondly, let us prove that $\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}$ converge uniformly towards $\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}$ over $\overline{R}^\lambda_{\mathbf{x}_0}$. Given that for all $\widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_0}$, $X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)) = \mathbf{y}$, we get

$$\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)) \times \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda) = O_{j,d-j}.$$

By this last fact, (3.7) and (3.8), we have the following sequence of inequalities

$$||\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda) - \nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)||_{d,d-j} =$$

$$||-(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda))/V_\lambda^\perp)^{-1} \times \left(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) - \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda))\right) \times \nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_{\lambda)}||_{d,d-j} \leqslant$$

$$||(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda))/V_\lambda^\perp)^{-1}||_{d,j} \times ||\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) - \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda))||_{j,d} \times ||\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)||_{d,d-j} \leqslant$$

$$\sup_{\mathbf{z}\in K} ||(\nabla X(\mathbf{z})/V_\lambda^\perp)^{-1}||_{d,j} \sup_{\widehat{\mathbf{x}}_\lambda \in \overline{R}^\lambda_{\mathbf{x}_0}} ||\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}(\widehat{\mathbf{x}}_\lambda)) - \nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda))||_{j,d} \times \sup_{\widehat{\mathbf{x}}_\lambda \in \overline{R}^\lambda_{\mathbf{x}_0}} ||\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}(\widehat{\mathbf{x}}_\lambda)||_{d,d-j}$$

The first term in the last inequality is bounded as we have explained above. The third term is also bounded because $\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}$ is continuous over $\overline{R}^\lambda_{\mathbf{x}_0}$ that is a compact. In the same form the second term tends to zero because we already know that $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0,\delta}}$ uniformly converges towards $\overrightarrow{\alpha_{\lambda,\mathbf{x}_0}}$ on $\overline{R}^\lambda_{\mathbf{x}_0}$ and ∇X is continuous over K a compact set.
We will show that

$$\lim_{\delta\to 0}\int_{\mathcal{C}_X^{D^r}(\mathbf{y}+\delta)} Y(\mathbf{z})d\sigma_{d-j}(\mathbf{z}) = \int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{z})d\sigma_{d-j}(\mathbf{z}).$$

Fisrtly using that $\text{supp}(Y)$ is a compact set in $\mathbb{R}^d$ included in the open set $\Gamma(\lambda)$, we can prove that there exists an open set O contained in $\mathbb{R}^d$ such that: $\text{supp}(Y) \subset O \subset \overline{O} \subset \Gamma(\lambda)$.
Secondly let us notice that since $\overline{O} \subset \Gamma(\lambda) \subset D$, the set $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$ is a compact set of $\mathbb{R}^d$.
Let us built a partition of unity $\{\pi_1, \cdots, \pi_m\}$ for this compact manifold in the following form.
Let us consider $\mathbf{x} \in \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$. Since $\mathbf{x} \in \overline{O} \subset \Gamma(\lambda)$, $\mathbf{x} \in \Gamma(\lambda)$ it holds that $J_X^{(\lambda)}(\mathbf{x}) \neq 0$ and we can construct the open set $U^\lambda_{\mathbf{x}}$. Since $U^\lambda_{\mathbf{x}}$ is open, we can chose a real number $r^\lambda_{\mathbf{x}} > 0$ such that the closed ball of $\mathbb{R}^d$ with center $\mathbf{x}$ and radius $r^\lambda_{\mathbf{x}}$ is contained in $U^\lambda_{\mathbf{x}}$, let $\overline{B}(\mathbf{x}, r^\lambda_{\mathbf{x}}) \subset U^\lambda_{\mathbf{x}}$.
But we know that $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$ is a compact set of $\mathbb{R}^d$, then we can cover $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$ with a finite number of these balls, that is $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \subset \cup_{i=1}^m B(\mathbf{x}_i, r^\lambda_{\mathbf{x}_i})$ in such a form that for $i = 1, \cdots, m$, we still have:

$$\overline{B}(\mathbf{x}_i, r^\lambda_{\mathbf{x}_i}) \subset U^\lambda_{\mathbf{x}_i}.$$

In the same form that in page 34, we obtain a partition of unity of $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$, let $\{\pi_1, \cdots, \pi_m\}$, such that:

1. π_i is C^∞ on $\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$, $\pi_i \geqslant 0$, $i = 1, \cdots, m$.

2. $\text{supp}(\pi_i) \subset \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \cap \overline{B}(\mathbf{x}_i, r^\lambda_{\mathbf{x}_i}) \subset U^\lambda_{\mathbf{x}_i}$.

3. $\sum_{i=1}^m \pi_i(\mathbf{x}) = 1$, $\mathbf{x} \in \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$.

Moreover, the sequence of inclusions holds true
$\text{supp}(Y) \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \subset O \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \subset \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \subset \cup_{i=0}^m U^\lambda_{\mathbf{x}_i}$, $\mathbf{x}_i \in \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$, $i = 1, \cdots, m$.
Let us use the fact that $\text{supp}(Y)$ is a compact set of $\mathbb{R}^d$ and O is open, in the following form. Consider $\omega \in \text{supp}(Y)$. Given that $\text{supp}(Y) \subset O \subset \Gamma(\lambda)$, there exists two open sets $\widetilde{U}(\omega)$ and $U(\omega)$ containing $\{\omega\}$ and $R_\omega > 0$, $\widetilde{U}(\omega) \subset U(\omega) \subset O$ such that the restriction $f_\lambda|_{\widetilde{U}(\omega)}$ has an inverse on the open ball $B(f_\lambda(\omega), R_\omega/2)$ and the restriction $f_\lambda|_{U(\omega)}$ has an inverse on the open ball $B(f_\lambda(\omega), R_\omega)$. Also it is possible to have $\text{Diam}(U(\omega)) \leqslant \inf_{i=1,\cdots,m} \text{Diam}(\widetilde{W}^\lambda_{\mathbf{x}_i})$, $\widetilde{W}^\lambda_{\mathbf{x}_i}$ being the neighborhood of zero in $\mathbb{R}^j$ used to build the function $\gamma_{\lambda,\mathbf{x}_i}$ and also the local parametrization of the level curve at level $\mathbf{y} + \delta$.
Since $\text{supp}(Y)$ is compact in $\mathbb{R}^d$, it can be cover by a finite number of such sets, that is $\text{supp}(Y) \subset \cup_{\ell=1}^k \widetilde{U}(\omega_\ell) \subset \cup_{\ell=1}^k U(\omega_\ell) \subset O$, $\omega_\ell \in \text{supp}(Y)$ for $\ell = 1, \cdots, k$.
Let us chose $\delta = (\delta_1, \cdots, \delta_j) \in \mathbb{R}^j$ such that $||\delta||_j \leqslant \inf_{\ell=1,\cdots,k} R_{\omega_\ell}/2$.
We are going to prove that if δ is small enough, all element of $\mathcal{C}_X^{D^r}(\mathbf{y} + \delta) \cap \text{supp}(Y)$ belongs to $\overrightarrow{\alpha_{\lambda,x_i,\delta}}(R^\lambda_{\mathbf{x}_i})$, for almost one indice i belonging to $1, \cdots, m$.
Let us chose $\mathbf{z} \in \mathcal{C}_X^{D^r}(\mathbf{y} + \delta) \cap \text{supp}(Y)$. Since $\mathbf{z} \in \text{supp}(Y)$, there exists $\ell = 1, \cdots, k$, such that $\mathbf{z} \in \widetilde{U}(\omega_\ell)$, and given that $\mathbf{z} \in \mathcal{C}_X^{D^r}(\mathbf{y} + \delta)$,

$f_\lambda(\mathbf{z}) = \pi_{V_\lambda}(\mathbf{z}) + \sum_{k=1}^{j} \delta_k \, e_{\ell_k} \in B(f_\lambda(\omega_\ell), R_{\omega_\ell}/2)$.
It yields that $\pi_{V_\lambda}(\mathbf{z}) \in B(f_\lambda(\mathbf{z}), ||\delta||_j) \subset B(f_\lambda(\omega_\ell), R_{\omega_\ell})$. Thus there exists an unique $\mathbf{x} \in U(\omega_\ell) \subset O$, such that

$f_\lambda(\mathbf{x}) = \pi_{V_\lambda}(\mathbf{z}) = \pi_{V_\lambda}(\mathbf{x}) + \sum_{k=1}^{j} (X_k(\mathbf{x}) - y_k) e_{\ell_k}$. So we have $\pi_{V_\lambda}(\mathbf{z}) = \pi_{V_\lambda}(\mathbf{x})$ and $\sum_{k=1}^{j} (X_k(\mathbf{x}) - y_k) e_{\ell_k} = 0$, thus $X(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{x} \in O \cap$

$C_X^{D^r}(\mathbf{y}) \subset \cup_{i=0}^m U_{\mathbf{x}_i}^\lambda$, $\mathbf{x}$ can be written in the form $\mathbf{x} = \overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{x}}_\lambda)$, $\widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_i}^\lambda$, $i = 1, \cdots, m$. Finally and since $\pi_{V_\lambda}(\mathbf{z}) = \pi_{V_\lambda}(\mathbf{x})$, the vector $\mathbf{z}$ can be written in the form $\mathbf{z} = \mathbf{x} + \pi_{V_\lambda^\perp}(\mathbf{z} - \mathbf{x})$. Moreover, $\pi_{V_\lambda^\perp}(\mathbf{z} - \mathbf{x}) = \sum_{k=1}^j \gamma_k e_{\ell_k}$ and $||\gamma||_j = ||\pi_{V_\lambda^\perp}(\mathbf{z} - \mathbf{x})||_d \leqslant ||\mathbf{z} - \mathbf{x}||_d \leqslant \mathrm{Diam}(U(\omega_\ell)) \leqslant \mathrm{Diam}(\widetilde{W}_{\mathbf{x}_i}^\lambda)$, because we have

$$\sup_{\ell=1,\cdots,k} \mathrm{Diam}(U(\omega_\ell)) \leqslant \inf_{i=1,\cdots,m} \mathrm{Diam}(\widetilde{W}_{\mathbf{x}_i}^\lambda).$$

Finally by using the property 3. of function G, we have proven that $\mathbf{z} = \overrightarrow{\alpha_{\lambda,x_i,\delta}}(\widehat{\mathbf{x}}_\lambda), \widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_i}^\lambda, i = 1, \cdots, m$.
Let us assume that $\mathbf{t}$, $\mathbf{s} \in (\cup_{i=0}^m U_{\mathbf{x}_i}^\lambda) \cap C_X^{D^r}(\mathbf{y})$ are such that $\pi_{V_\lambda}(\mathbf{t}) = \pi_{V_\lambda}(\mathbf{s})$ and $\mathbf{t} \neq \mathbf{s}$. We can write $\mathbf{s} = (\pi_{V_\lambda^\perp}(\mathbf{t}) + \sigma_0\gamma) + \pi_{V_\lambda}(\mathbf{t})$, where $\sigma_0 > 0$, $\gamma \in V_\lambda^\perp$ and $||\gamma||_d = 1$. By defining $h(\sigma) = X(\mathbf{t} + \sigma\gamma)$ we have $h(0) = h(\sigma_0) = \mathbf{y}$. Let us point out that this function is well defined for $0 \leqslant \sigma \leqslant \sigma_0$, because D is a convex open set and here is the only place where we use the convexity of the set D . Also, the Rolle's theorem allows us to say that if $h = (h_1, \cdots, h_j)$, then for all $\ell = 1, \cdots, j$, there exists $\sigma_\ell \in (0, \sigma_0)$ such that $\dot{h}_\ell(\sigma_\ell) = \nabla X_\ell(\mathbf{t} + \sigma_\ell\gamma)(\gamma) = 0$. Moreover,

$$1 = ||\gamma||_d = ||(\nabla X(\mathbf{t})|_{V_\lambda^\perp})^{-1}(\nabla X(\mathbf{t})|_{V_\lambda^\perp}(\gamma))||_d \leqslant M||\nabla X(\mathbf{t})(\gamma)||_j,$$

where $M = \sum_{i=1}^m \sup_{\mathbf{x}\in\overline{U}_{\mathbf{x}_i}^\lambda} ||(\nabla X(\mathbf{x})|_{V_\lambda^\perp})^{-1}||_{d,j} < +\infty$ (to ensure that the last norm is finite it is sufficient to take the open sets $U_{\mathbf{x}_i}^\lambda$ small enough). Finally by using that $\dot{h}_\ell(\sigma_\ell) = 0$ we obtain

$$\begin{aligned} 1 &\leqslant M^2 \sum_{\ell=1}^j |\nabla X_\ell(\mathbf{t} + \sigma_\ell\gamma)(\gamma) - \nabla X_\ell(\mathbf{t})(\gamma)|^2 \\ &\leqslant M^2 \sum_{\ell=1}^j ||\nabla X_\ell(\mathbf{t} + \sigma_\ell\gamma) - \nabla X_\ell(\mathbf{t})||_{1,d}^2. \end{aligned}$$

We will asume for the first time that ∇X is a Lipschitz function. Let L be its Lipschitz's constant. We have

$$1 \leqslant M^2 L^2 \sum_{\ell=1}^j \sigma_\ell^2 \leqslant j M^2 L^2 \sigma^2,$$

where $\sigma = \max_{\ell=1,\cdots,j} \sigma_\ell$. From this fact it yields

$$||\mathbf{s}-\mathbf{t}||_d = \sigma_0 \geqslant \sigma \geqslant \frac{1}{\sqrt{j}ML} = a. \tag{3.10}$$

Let us prove that if $\mathbf{z} \in \mathcal{C}_X^{D^r}(\mathbf{y}+\delta) \cap \mathrm{supp}(Y)$, there exists a unique $\mathbf{x} \in \cup_{i=0}^m U_{\mathbf{x}_i}^\lambda \cap \mathcal{C}_X^{D^r}(\mathbf{y})$, such that $\mathbf{z} = \mathbf{x} + \pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x})$ and such that $||\pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x})||_d \leqslant \inf_{i=1,\cdots,m} \mathrm{Diam}(\widetilde{W}_{\mathbf{x}_i}^\lambda)$ (this is always possible by taking the open sets $\widetilde{W}_{\mathbf{x}_i}^\lambda$ small enough to have $\mathrm{Diam}(\widetilde{W}_{\mathbf{x}_i}^\lambda) < \dfrac{a}{2}$).
As $\mathbf{z}$ belongs to $\mathcal{C}_X^{D^r}(\mathbf{y}+\delta) \cap \mathrm{supp}(Y)$, we have shown above the existence of this $\mathbf{x}$. Let us show now the unicity. We assume that there exists another vector $\mathbf{x}' \in (\cup_{i=0}^m U_{\mathbf{x}_i}^\lambda) \cap \mathcal{C}_X^{D^r}(\mathbf{y})$, $\mathbf{x}' \neq \mathbf{x}$, such that $\mathbf{z} = \mathbf{x}' + \pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x}')$ and also that $||\pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x}')||_d \leqslant \inf_{i=1,\cdots,m} \mathrm{Diam}(\widetilde{W}_{\mathbf{x}_i}^\lambda)$.
For the precedent result we have necessarily $||\mathbf{x}-\mathbf{x}'||_d \geqslant a$.
Furthermore, it holds

$$\begin{aligned} ||\mathbf{x}-\mathbf{x}'||_d &\leqslant ||\mathbf{x}-\mathbf{z}||_d + ||\mathbf{x}'-\mathbf{z}||_d \\ &= ||\pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x})||_d + ||\pi_{V_\lambda^\perp}(\mathbf{z}-\mathbf{x}')||_d \\ &< \frac{a}{2} + \frac{a}{2} = a. \end{aligned}$$

Thus we get a contradiction.
Now consider $\mathbf{z} \in \mathcal{C}_X^{D^r}(\mathbf{y}+\delta) \cap \mathrm{supp}(Y)$, since

$$\mathbf{x} \in O \cap \mathcal{C}_X^{D^r}(\mathbf{y}) \subset \overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})$$

and given that $\widehat{\mathbf{z}}_\lambda = \widehat{\mathbf{x}}_\lambda$, we obtain the sequence of equalities

$$\begin{aligned} Y(\mathbf{z}) &= \left(\sum_{i=1}^m \pi_i(\mathbf{x})\right) \times Y(\mathbf{z}) \\ &= \sum_{i=1}^m \pi_i(\mathbf{x}) \times 1_{\{\mathbf{x} \in U_{\mathbf{x}_i}^\lambda \cap \mathcal{C}_X^{D^r}(\mathbf{y})\}} \times Y(\mathbf{z}) \\ &= \sum_{i=1}^m \pi_i(\mathbf{x}) \times 1_{\{\mathbf{x} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)\}} \times 1_{\{\widehat{\mathbf{x}}_\lambda \in R_{\mathbf{x}_i}^\lambda\}} \times Y(\mathbf{z}) \\ &= \sum_{i=1}^m \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{z}}_\lambda)) \times 1_{\{\mathbf{x} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{z}}_\lambda)\}} \times 1_{\{\widehat{\mathbf{z}}_\lambda \in R_{\mathbf{x}_i}^\lambda\}} \times Y(\mathbf{z}) \end{aligned}$$

$$= \sum_{i=1}^{m} \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{z}}_\lambda)) \times 1_{\{\mathbf{z}=\overrightarrow{\alpha_{\lambda,\mathbf{x}_i,\delta}}(\widehat{\mathbf{z}}_\lambda)\}} \times Y(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i,\delta}}(\widehat{\mathbf{z}}_\lambda)) \times 1_{\{\widehat{\mathbf{z}}_\lambda \in R^\lambda_{\mathbf{x}_i}\}}$$

The last equality coming from the way we built the vector $\mathbf{x}$ from the vector $\mathbf{z}$ and from the uniqueness of the decomposition of this vector $\mathbf{z}$ on the set $(\cup_{i=0}^{m} U^\lambda_{\mathbf{x}_i}) \cap \mathcal{C}_X^{D^r}(\mathbf{y})$.
We have the following equality

$$\begin{aligned}
&\int_{\mathcal{C}_X^{D^r}(\mathbf{y}+\delta)} Y(\mathbf{z}) d\sigma_{d-j}(\mathbf{z}) \\
= &\sum_{i=1}^{m} \int_{\{\mathbf{z}=\overrightarrow{\alpha_{\lambda,\mathbf{x}_i,\delta}}(\widehat{\mathbf{z}}_\lambda),\, \widehat{\mathbf{z}}_\lambda \in R^\lambda_{\mathbf{x}_i}\}} \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{z}}_\lambda)) \times Y(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i,\delta}}(\widehat{\mathbf{z}}_\lambda)) d\sigma_{d-j}(\mathbf{z}) =
\end{aligned}$$

$$\sum_{i=1}^{m} \int_{R^\lambda_{\mathbf{x}_i}} \pi_i(\overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)) Y(\overrightarrow{\alpha_{\lambda,x_i,\delta}}(\widehat{\mathbf{z}}_\lambda)) (\det(\nabla \overrightarrow{\alpha_{\lambda,x_i,\delta}}(\widehat{\mathbf{z}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,x_i,\delta}}(\widehat{\mathbf{z}}_\lambda)^T))^{1/2} \, \mathrm{d}\widehat{\mathbf{z}}_\lambda$$

$$\to \sum_{i=1}^{m} \int_{R^\lambda_{\mathbf{x}_i}} \pi_i(\overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)) Y(\overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)) (\det(\nabla \overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)^T))^{1/2} \, \mathrm{d}\widehat{\mathbf{z}}_\lambda,$$

the last convergence comes from the uniform convergence of $\overrightarrow{\alpha_{\lambda,\mathbf{x}_i,\delta}}$ to $\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}$ over $\overline{R}^\lambda_{\mathbf{x}_i}$ and of that one of $\nabla \overrightarrow{\alpha_{\lambda,x_i,\delta}}$ towards $\nabla \overrightarrow{\alpha_{\lambda,x_i}}$ over $\overline{R}^\lambda_{\mathbf{x}_i}$.
But

$$\sum_{i=1}^{m} \int_{R^\lambda_{\mathbf{x}_i}} \pi_i(\overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)) Y(\overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)) (\det(\nabla \overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,x_i}}(\widehat{\mathbf{z}}_\lambda)^T))^{1/2} \, \mathrm{d}\widehat{\mathbf{z}}_\lambda$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \int_{U^\lambda_{\mathbf{x}_i} \cap \mathcal{C}_X^{D^r}(\mathbf{y})} \pi_i(\mathbf{x}) Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}) = \int_{\overline{O} \cap \mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}) \\
&\qquad = \int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}),
\end{aligned}$$

because $\mathrm{supp}(Y) \subset \overline{O}$.
Summing up we have proved the continuity of the function

$$\mathbf{y} \to \int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) \, \mathrm{d}\sigma_{d-j}(\mathbf{x}),$$

under the hypothesis that $Y : D^r_X \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function satisfying $\mathrm{supp}(Y) \subset \Gamma(\lambda)$.

Finally, we do not assume anymore that $\text{supp}(Y) \subset \Gamma(\lambda)$, just that $\text{supp}(Y) \subset D_X^r$.
Let us introduce two new functions. For $\lambda \in B_j$, let set for $\mathbf{t} \in D$,

$$\phi_\lambda(\mathbf{t}) = \inf_{\gamma \in V_\lambda^\perp, ||\gamma||_d = 1} ||\nabla X(\mathbf{t})/_{V_\lambda^\perp}(\gamma)||_j, \tag{3.11}$$

and

$$\phi(\mathbf{t}) = \sup_{\lambda \in B_j} \phi_\lambda(\mathbf{t}).$$

These two functions are Lipschitz then continuous, with the same Lipschitz constant L, that the one of ∇X. In fact, let us consider firstly the first function. Consider two points $\mathbf{t}$ and $\mathbf{t}^\star$, we have for all $\gamma \in V_\lambda^\perp$ satisfying $||\gamma||_d = 1$,

$$\begin{aligned} ||\nabla X(\mathbf{t})/_{V_\lambda^\perp}(\gamma)||_j &= ||\nabla X(\mathbf{t})(\gamma)||_j \\ \leqslant ||\nabla X(\mathbf{t})(\gamma) - \nabla X(\mathbf{t}^*)(\gamma)||_j &+ ||\nabla X(\mathbf{t}^*)/_{V_\lambda^\perp}(\gamma)||_j, \end{aligned}$$

using that ∇X is Lipschitz, we obtain

$$||\nabla X(\mathbf{t})/_{V_\lambda^\perp}(\gamma)||_j \leqslant L||\mathbf{t} - \mathbf{t}^\star||_d + ||\nabla X(\mathbf{t}^*)/_{V_\lambda^\perp}(\gamma)||_j,$$

then

$$\phi_\lambda(\mathbf{t}) \leqslant L||\mathbf{t} - \mathbf{t}^\star||_d + \phi_\lambda(\mathbf{t}^*), \tag{3.12}$$

since a symmetric inequality can be proven, we obtain finally

$$|\phi_\lambda(\mathbf{t}) - \phi_\lambda(\mathbf{t}^*)| \leqslant L||\mathbf{t} - \mathbf{t}^\star||_d.$$

Let us study now the second function.
The inequality (3.12) allows writing

$$\phi(\mathbf{t}) \leqslant L||\mathbf{t} - \mathbf{t}^\star||_d + \phi(\mathbf{t}^*),$$

given in the same form as before

$$|\phi(\mathbf{t}) - \phi(\mathbf{t}^*)| \leqslant L||\mathbf{t} - \mathbf{t}^\star||_d.$$

Let us prove that

$$\left(\lambda \in B_j \text{ and } \mathbf{t} \in \Gamma(\lambda)\right) \iff (\phi_\lambda(\mathbf{t}) > 0)\,. \tag{3.13}$$

Let us consider $\mathbf{t} \in \Gamma(\lambda)$ for a $\lambda \in B_j$. That means that $\nabla X(\mathbf{t})/_{V_\lambda^\perp}$ has an inverse and moreover that $\ker(\nabla X(\mathbf{t})/_{V^\perp}) = \vec{0}/_{V^\perp}$. But there exists a $\gamma_0 \in V_\lambda^\perp$, $||\gamma_0||_d = 1$ such that $\phi_\lambda(\mathbf{t}) = ||\nabla X(\mathbf{t})/_{V_\lambda^\perp}(\gamma_0)||_j$, and this implies that $\phi_\lambda(\mathbf{t}) > 0$.
To prove the other implication, let assume that for a $\lambda \in B_j$, $\mathbf{t} \notin \Gamma(\lambda)$, say that $\nabla X(\mathbf{t})/_{V_\lambda^\perp}$ has not inverse. Then there exists $\gamma \in V_\lambda^\perp$, $||\gamma||_d = 1$ such that $\nabla X(\mathbf{t})/_{V_\lambda^\perp}(\gamma) = 0$, and this implies $\phi_\lambda(\mathbf{t}) = 0$.
Now let us prove that

$$(\mathbf{t} \in D_X^r) \iff (\phi(\mathbf{t}) > 0) \tag{3.14}$$

Indeed by Remark 3.1.3 and equivalence (3.13), we have the following equivalences

$$\begin{aligned}(\mathbf{t} \in D_X^r) \iff \left(\exists\, \lambda \in B_j,\, \mathbf{t} \in \Gamma(\lambda)\right) \quad &\iff \quad \left(\exists\, \lambda \in B_j,\, \phi_\lambda(\mathbf{t}) > 0\right) \\ &\iff \quad (\phi(\mathbf{t}) > 0)\end{aligned}$$

We are going to built a partition of unity of D_X^r whose support intersected with D_X^r will be included in $\Gamma(\lambda)$, for all $\lambda \in B_j$. We will denote this partition by η_λ .
Firstly let us consider the function $\chi_\lambda(\mathbf{t}) = (2\phi_\lambda(\mathbf{t}) - \phi(\mathbf{t}))^+$. Since ϕ_λ and ϕ are Lipschitz functions, it results that $2\phi_\lambda - \phi$ remains Lipschitz. It follows that the function χ_λ is also Lipschitz and a fortiori continuous.

Let us show that

$$(\mathbf{t} \in D_X^r) \Longrightarrow \left(\sum_{\lambda \in B_j} \chi_\lambda(\mathbf{t}) > 0\right) \tag{3.15}$$

Consider $\mathbf{t} \in D_X^r$. Let assume that $\sum_{\lambda \in B_j} \chi_\lambda(\mathbf{t}) = 0$ and let us prove that we get a contradiction. Since $\sum_{\lambda \in B_j} \chi_\lambda(\mathbf{t}) = 0$, for all $\lambda \in B_j$ we have then $\chi_\lambda(\mathbf{t}) = 0$, that is $\phi_\lambda(\mathbf{t}) \leqslant \frac{1}{2}\phi(\mathbf{t})$. This implies because this inequality holds true for all $\lambda \in B_j$, that $\phi(\mathbf{t}) \leqslant \frac{1}{2}\phi(\mathbf{t})$ and $\phi(\mathbf{t}) = 0$, which is in contradiction with the equivalence (3.14).

Let set for all $\mathbf{t} \in D_X^r$,

$$\eta_\lambda(\mathbf{t}) = \frac{\chi_\lambda(\mathbf{t})}{\sum\limits_{\lambda \in B_j} \chi_\lambda(\mathbf{t})}, \tag{3.16}$$

that is possible from property (3.15). Now it is easy to see that η_λ is continuous on D_X^r since χ_λ is also continuous on D_X^r.
It remains only to prove that the support of this function intersected with D_X^r is included in $\Gamma(\lambda)$.
For all $C > 0$, let us build an open set O_C around of

$$(\Gamma(\lambda))^{c_1} \cap \{\mathbf{t} \in D, \phi(\mathbf{t}) > C\},$$

contained in the set

$$\{\mathbf{t} \in D, \chi_\lambda(\mathbf{t}) = 0\}.$$

More precisely, let us prove that for a given $C > 0$, if $\delta \leqslant \dfrac{C}{2L}$ (with L being the Lipschitz constant of the function ∇X), then

$$((\Gamma(\lambda))^{c_1})_\delta \cap \{\mathbf{t} \in D :\, \phi(\mathbf{t}) > C\} \subset \{\mathbf{t} \in D :\, \chi_\lambda(\mathbf{t}) = 0\}.$$

where for all set A we have defined the open set $A_\delta = \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, A) < \delta\}$.
Thus let $C > 0$ be fixed and $\mathbf{t} \in ((\Gamma(\lambda))^{c_1})_\delta \cap \{\mathbf{t} \in D :\, \phi(\mathbf{t}) > C\}$. Since $\mathbf{t} \in ((\Gamma(\lambda))^{c_1})_\delta$, then there exists a $\mathbf{t}' \in B(\mathbf{t}, \delta)$ such that $\mathbf{t}' \in (\Gamma(\lambda))^{c_1}$ (and also such that $\phi_\lambda(\mathbf{t}') = 0$ as a consequence of the implication (3.13)). Then we have, since ϕ_λ is Lipschitz with Lipschitz constant L that

$$\begin{aligned} \phi_\lambda(\mathbf{t}) &\leqslant |\phi_\lambda(\mathbf{t}) - \phi_\lambda(\mathbf{t}')| + \phi_\lambda(\mathbf{t}') \\ &\leqslant L||\mathbf{t} - \mathbf{t}'||_d \leqslant L\,\delta \leqslant \frac{C}{2}, \end{aligned}$$

hence $2\phi_\lambda(\mathbf{t}) \leqslant C < \phi(\mathbf{t})$, and this entails that $\chi_\lambda(\mathbf{t}) = 0$. We have proved that for all $C > 0$ and for all $\delta \leqslant \frac{C}{2L}$, we have the inclusion

$$\begin{aligned} &\{\mathbf{t} \in D_X^r :\, \chi_\lambda(\mathbf{t}) \neq 0\} = \{\mathbf{t} \in D_X^r :\, \eta_\lambda(\mathbf{t}) \neq 0\} \subset \\ &\quad ((((\Gamma(\lambda))^{c_1})_\delta)^c \cap D_X^r) \cup \{\mathbf{t} \in D_X^r :\, \phi(\mathbf{t}) \leqslant C\}, \end{aligned}$$

where we recall that the symbol c denotes the complementary set with respect to $\mathbb{R}^d$.
Noting that $(((\Gamma(\lambda))^{c_1})_\delta)^c$ is a closed set contained in $\Gamma(\lambda) \cup D^c$, we have

$$\mathrm{supp}(\eta_\lambda) \subset [(\Gamma(\lambda) \cup D^c) \cap \overline{D^r_X}] \cup (\cap_{C>0}\overline{\{\mathbf{t} \in D^r_X : \phi(\mathbf{t}) \leqslant C\}}),$$

that is

$$\mathrm{supp}(\eta_\lambda) \cap D^r_X \subset \Gamma(\lambda) \cup (\cap_{C>0}\overline{\{\mathbf{t} \in D^r_X : \phi(\mathbf{t}) \leqslant C\}} \cap D^r_X).$$

It will be enough to finish to prove that $\cap_{C>0}\overline{\{\mathbf{t} \in D^r_X : \phi(\mathbf{t}) \leqslant C\}} \cap D^r_X = \emptyset$. Indeed, consider $\mathbf{z} \in \cap_{C>0}\overline{\{\mathbf{t} \in D^r_X : \phi(\mathbf{t}) \leqslant C\}} \cap D^r_X$. Then $\mathbf{z} \in D^r_X$ and for all $C > 0$, there exists a sequence of points $\mathbf{z}_{n,C}$ of D^r_X, satisfying $\phi(\mathbf{z}_{n,C}) \leqslant C$ and that converges to $\mathbf{z} \in D^r_X$. Since the function ϕ is continuous on D and also on D^r_X, it holds that $\phi(\mathbf{z}) \leqslant C$. This last inequality is true for all $C > 0$, then we get that $\phi(\mathbf{z}) = 0$. From property (3.14) we readily get that $\mathbf{z} \in (D^r_X)^{c_1}$. But $\mathbf{z} \in D^r_X$.
We have proved that

$$\mathrm{supp}(\eta_\lambda) \cap D^r_X \subset \Gamma(\lambda). \tag{3.17}$$

In this form we have for $\mathbf{t} \in D^r_X$, $Y(\mathbf{t}) = \sum_{\lambda \in B_j} \eta_\lambda(\mathbf{t})Y(\mathbf{t}) = \sum_{\lambda \in B_j} Y_\lambda(\mathbf{t})$, where we have set for $\mathbf{t} \in D^r_X$, $Y_\lambda(\mathbf{t}) = \eta_\lambda(\mathbf{t})Y(\mathbf{t})$.
The function Y_λ is a continuous function on D^r_X with compact support included in $\Gamma(\lambda)$, from the inclusion (3.17) and since $\mathrm{supp}(Y) \subset D^r_X$ by hypothesis.
We have for all $\mathbf{y} \in \mathbb{R}^j$

$$\int_{\mathcal{C}^{D^r}_X(\mathbf{y})} Y(\mathbf{z})d\sigma_{d-j}(\mathbf{z}) = \sum_{\lambda \in B_j} \int_{\mathcal{C}^{D^r}_X(\mathbf{y})} Y_\lambda(\mathbf{z})d\sigma_{d-j}(\mathbf{z}),$$

the continuity of the left hand side integral as a function of the $\mathbf{y}$ variable is a consequence of the continuity of each of the terms of the sum in the right hand side. This last fact is an application of the above procedure.
This finish the proof of Theorem 3.1.2 in the case where we have chosen the bounded open set D_1 of $\mathbb{R}^d$ equal to D convex (bounded).

Let assume now that D is a convex open set of $\mathbb{R}^d$, that can be unbounded. The function $X : D \subset \mathbb{R}^d \to \mathbb{R}^j$ is a function $\mathbf{C}^1(D, \mathbb{R}^j)$ such that ∇X is Lipschitz and the function $Y : D_1 \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function defined on D_1 open and bounded set of $\mathbb{R}^d$ included in D such that $\text{supp}(Y) \subset D^r_{X/D_1}$.
In this case the function X restricted to the bounded open set D_1, that is X/D_1, is such that $X/D_1 : D_1 \subset \mathbb{R}^d \to \mathbb{R}^j$ is still a continuous function $\mathbf{C}^1(D_1, \mathbb{R}^j)$ such that $\nabla X/D_1$ is Lipschitz.
One can apply the precedent procedure to these two functions X/D_1 and Y and also to the open set D_1. The only problematic thing is that a priori the set D_1 could not be convex, but this is not a true problem. Indeed, if we refer to page 40, the only place where we used the convexity of the open set, we realize that what is important is to be able applying the Rolle's theorem to the function h that is defined there and using the fact that the function ∇X is Lipschitz. Since D_1 could not be convex, we did not be sure to can do that, but this is not the case if one works on D that is convex.
Ending the proof of the theorem. □

Now we are able to exhibit a class of processes X and Y satisfying the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ through the following condition $\mathbf{A_0}$ and the following proposition whose proof is based on the one given by Cabaña [11]. In what follows we will give a new proof slightly more general that the original one.

- $\mathbf{A_0}$: $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is a random field that belongs to $\mathbf{C}^1(D, \mathbb{R}^j)$, where D is a bounded open convex set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, the process $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ satisfying $\mathbb{E}(L_X(\cdot))^d < +\infty$. Also $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a continuous process such that there exists a $\lambda \in B_j$ such that $\text{supp}(Y) \subset \Gamma(\lambda)$. Moreover, $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$, $Y(\cdot)$ and $||\nabla X(\cdot)||_{j,d}$ are assumed uniformly bounded on the support of Y, the bounds not depending on ω ($\in \Omega$).

Proposition 3.1.1 *If X and Y satisfy the condition $\mathbf{A_0}$, then the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ are satisfied.*

Remark 3.1.5 We can replace in $\mathbf{A_0}$ the condition $\mathbb{E}(L_X(\cdot))^d < +\infty$ for the following, there exists a $L > 0$ such that for almost surely $\omega \in \Omega$, the hypotheses $\text{supp}(Y(\omega)) \neq \emptyset$ implies that $L_X(\omega) \leqslant L$, then the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ hold.

Remark 3.1.6 We can generalize the Proposition 3.1.1 and also the Remark 3.1.5 assuming that D is an open and convex set eventually unbounded and maintaining the same hypotheses on X. Furthermore for Y we will assume that it is defined on D_1 bounded and open set included in D. Moreover, we need to adapt the hypotheses for Y to the open set D_1 in place of D and to X/D_1. The hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ will be still hold for X/D_1 and Y defined on D_1.

Proof of the proposition 3.1.1. For almost surely $\omega \in \Omega$ the field $X(\omega) : D \subset \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) belongs to $\mathbf{C}^1(D, \mathbb{R}^j)$ such that $\nabla X(\omega)$ is Lipschitz and $Y(\omega) : D \subset \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that $\text{supp}(Y(\omega)) \subset \Gamma(\lambda)(\omega) \subset D^r_{X(\omega)}$ and D is an open and convex bounded set of $\mathbb{R}^d$. According to the Theorem 3.1.2 the function

$$\mathbf{y} \longmapsto \int_{\mathcal{C}^{D^r}_{X(\omega)}(\mathbf{y})} Y(\omega)(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})$$

is a continuous function of the variable $\mathbf{y}$.
The same is true for $\int_{\mathcal{C}^{D^r}_{X(\omega)}(\mathbf{y})} |Y(\omega)(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})$.
Let us bound by above $\int_{\mathcal{C}^{D^r}_{X}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})$ by an integrable random variable that does not depend on $\mathbf{y}$. Then according to the dominated convergence theorem hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ will be fullfilled.
Since $\text{supp}(Y) \subset \Gamma(\lambda)$, we can built a partition of unity of $\text{supp}(Y)$ in the same form as in page 34 getting as in (3.4)

$$\int_{\mathcal{C}^{D^r}_{X}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}) =$$

$$\sum_{i=1}^{m} \int_{R^\lambda_{\mathbf{x}_i}} \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)) Y(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)) (\det(\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)^T))^{1/2} \, d\widehat{\mathbf{x}}_\lambda.$$

Consider $\widehat{\mathbf{x}}_\lambda$ fixed in $R^\lambda_{\mathbf{x}_i}$ such that $\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \in \text{supp}(Y)$, $i = 1, \ldots, m$.
We have $\det(\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)^T))^{1/2} \leqslant ||\nabla \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))||^d_{d,d-j}$.

Let us bound uniformly $||\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))||_{d,d-j}$, for all $\widehat{\mathbf{x}}_\lambda$ in $R^\lambda_{\mathbf{x}_i}$ such that $\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \in \text{supp}(Y)$.
For all $\widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_i}$, we have $X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)) = \mathbf{y}$. Taken derivatives in this equality on the open set $R^\lambda_{\mathbf{x}_i}$, we obtain

$$\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)) \times \nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) = 0.$$

By using the equality (3.3), also that $\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) \in \Gamma(\lambda)$, for all $\mathbf{u} \in \mathbb{R}^{d-j}$, $\mathbf{u} = (u_1, \cdots, u_{d-j})$ it yields

$$\begin{aligned}
&\nabla\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\mathbf{u}) \\
&= -\left[\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))/V_\lambda^\perp\right]^{-1}\left(\nabla X(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))/V_\lambda(\sum_{k=1}^{d-j} u_k e_{i_k})\right) \\
&+ \sum_{k=1}^{d-j} u_k e_{i_k}.
\end{aligned}$$

Since $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$, $Y(\cdot)$ and $||\nabla X(\cdot)||_{j,d}$ are uniformly bounded on the support of Y and the bound does not depend of ω, then we have

$$\begin{aligned}
&\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x}) \\
&\leqslant \mathbf{C}\int_{\Pi_{V_\lambda}(D)} \sum_{i=1}^{m} \pi_i(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))\mathbb{1}_{\{\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)\in\text{supp}(Y)\}}\,\mathbb{1}_{\{\widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_i}\}}\,\mathrm{d}\widehat{\mathbf{x}}_\lambda.
\end{aligned}$$

For $\boldsymbol{\omega} \in \Omega$ and $\widehat{\mathbf{x}}_\lambda \in \Pi_{V_\lambda}(D)$, we consider the set A defined by

$$\begin{aligned}
A = \{&\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}), \widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_i}(\boldsymbol{\omega}) \\
&\text{and } \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}) \in \text{supp}(Y)(\boldsymbol{\omega}), i = 1, \dots, m\}
\end{aligned}$$

We form a partition of set A into equivalence classes. An equivalence class A_{i_0} for $i_0 = 1, \dots, m$, is the set defined as

$$\begin{aligned}
A_{i_0} = \{&\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}), \widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_i}(\boldsymbol{\omega}) \cap R^\lambda_{\mathbf{x}_{i_0}}(\boldsymbol{\omega}) \text{ and } \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}) \\
&= \overrightarrow{\alpha_{\lambda,\mathbf{x}_{i_0}}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}) \in \text{supp}(Y)(\boldsymbol{\omega}), i = 1, \dots, m\}.
\end{aligned}$$

By property 3. page 34, we have the following property $\sum_{i=1}^{m} \pi_i(\mathbf{x}) = 1$, $\mathbf{x} \in \mathrm{supp}(Y)$. Also into each class we bound the corresponding sum by one. Now it only remains to count the maximal number of equivalence classes.
For counting the classes let us take two elements belonging to two different classes. For fixing the ideas we will take for instance

$$\mathbf{t} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}), \widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_i}(\boldsymbol{\omega}) \text{ and } \overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}) \in \mathrm{supp}(Y)(\boldsymbol{\omega})$$

and

$$\mathbf{s} = \overrightarrow{\alpha_{\lambda,\mathbf{x}_j}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}),\ \widehat{\mathbf{x}}_\lambda \in R^\lambda_{\mathbf{x}_j}(\boldsymbol{\omega}) \text{ and } \overrightarrow{\alpha_{\lambda,\mathbf{x}_j}}(\widehat{\mathbf{x}}_\lambda)(\boldsymbol{\omega}) \in \mathrm{supp}(Y)(\boldsymbol{\omega}),$$

$i, j = 1, \cdots, m$ and $\mathbf{t} \neq \mathbf{s}$.
It is clear that $\mathbf{t}$ and $\mathbf{s}$ are two different elements of $\mathbb{R}^d$, they have the same projection on V_λ and belong to the level curve $C_X^{D^r}(\mathbf{y})$. By repeating the proof given in page 40 and since $\mathbf{t} \in \mathrm{supp}(Y)(\boldsymbol{\omega})$ and that $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$ is uniformly bounded on $\mathrm{supp}(Y)$ by a constant $\mathbf{C}$, we have the following bound

$$1 \leqslant \quad \mathbf{C^2} \textstyle\sum_{\ell=1}^{j} ||\nabla X_\ell(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda) + \sigma_\ell \gamma)(\boldsymbol{\omega}) - \nabla X_\ell(\overrightarrow{\alpha_{\lambda,\mathbf{x}_i}}(\widehat{\mathbf{x}}_\lambda))(\boldsymbol{\omega})||^2_{1,d}.$$

But for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$, we obtain

$$1 \leqslant j\, \mathbf{C^2} L_X^2(\boldsymbol{\omega})\, \sigma^2(\boldsymbol{\omega}),$$

where $\sigma = \max_{\ell=1,\cdots,j} \sigma_\ell$.
As in inequality (3.10) we finally get the following bound

$$||\mathbf{s} - \mathbf{t}||_d = \sigma_0(\boldsymbol{\omega}) \geqslant \sigma(\boldsymbol{\omega}) \geqslant \frac{1}{\sqrt{j}\mathbf{C} L_X(\boldsymbol{\omega})} = a(\boldsymbol{\omega}).$$

The open ball with centers $\mathbf{t}$ and $\mathbf{s}$ and diameter $a(\boldsymbol{\omega})$ do not intersect. We have at most $(\frac{diam(D)}{a(\boldsymbol{\omega})})^d$ balls of diameter $a(\boldsymbol{\omega})$ and then at most $(\frac{diam(D)}{a(\boldsymbol{\omega})})^d$ equivalence classes.
Finally for almost surely $\boldsymbol{\omega} \in \Omega$:

$$\int_{\mathcal{C}^{D^r}_{X(\boldsymbol{\omega})}(\mathbf{y})} |Y(\boldsymbol{\omega})(\mathbf{x})| d\sigma_{d-j}(\mathbf{x}) \leqslant \mathbf{C} \int_{\Pi_{V_\lambda}(D)} (\frac{diam(D)}{a(\boldsymbol{\omega})})^d \; \mathrm{d}\widehat{\mathbf{x}}_\lambda$$
$$\leqslant \mathbf{C}\, \sigma_{d-j}(\Pi_{V_\lambda}(D))\, (diam(D)\sqrt{j}\mathbf{C})^d L_X^d(\boldsymbol{\omega}) \leqslant \mathbf{C}\, L_X^d(\boldsymbol{\omega}).$$

Since $\mathbb{E}(L_X(\cdot))^d < +\infty$, $\mathbf{y} \longmapsto \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]$ is continuous. The same holds true for $\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$.
Then the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$ have been checked. This ends the proof of the proposition. □

Proof of the Remark 3.1.5. It is enough to replace in the proof of the preceding proposition $L_X(\boldsymbol{\omega})$ by L and $a(\boldsymbol{\omega})$ by $\frac{1}{\sqrt{j\mathbf{C}L}}$. In this case for almost surely $\boldsymbol{\omega} \in \Omega$, we have the bound

$$\int_{\mathcal{C}_{X(\boldsymbol{\omega})}^{D^r}(\mathbf{y})} |Y(\boldsymbol{\omega})(\mathbf{x})| d\sigma_{d-j}(\mathbf{x}) \leqslant \mathbf{C}.$$

and this implies the integrability. □

Proof of the Remark 3.1.6. We prove this remark in the same way as in the proof of Proposition 3.1.1. As in the proof of Theorem 3.1.2 we use that the open set D_1 is contained in D that is convex. This allows us applying, as in page 40, the Rolle's theorem and the fact that ∇X is Lipschitz on D. □

We will study now the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ and will show the following proposition that is also deeply inspired by Cabaña [11].
In what follows we will exhibit a class of processes X and Y satisfying these hypotheses. So we are going to set out some conditions concerning the processes X and Y.
In the first three conditions $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$, we will make the hypothesis that Y can be written as a function G of X, ∇X and of a new variable $W : \Omega \times D \subset \mathbb{R}^d \to \mathbb{R}^k$, $k \in \mathbb{N}^\star$, where D is an open set of $\mathbb{R}^d$, in the following form: for almost surely $\mathbf{x} \in D$:

$$Y(\mathbf{x}) = G(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})), \tag{3.18}$$

where

$$G : D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j) \longrightarrow \mathbb{R}$$
$$(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \longmapsto G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}),$$

is a continuous function of their variables on $D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and such that $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant P(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}),$$

where P is a polynomial having positive coefficients and $f : D \longrightarrow \mathbb{R}^+$ and $h : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ are continuous functions.

- **A$_1$**: The process $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is Gaussian belonging to C^1 on D, such that there exists a real a, $0 < a$, such that for almost surely $\mathbf{x} \in D$, $0 < a \leqslant \inf_{||z||_j=1} ||\mathbb{V}(X(\mathbf{x})) \times z||_j$. Also the first order partial derivatives of its covariance Γ_X are bounded almost surely over the diagonal contained in $D \times D$. Moreover, for almost surely $\mathbf{x} \in D$, the process $W(\mathbf{x})$ is independent of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$, and $\forall\, p \in \mathbb{N}$, $\forall\, n \in \mathbb{N}$, $\forall\, \ell \in \mathbb{N}$ and $\forall\, m \in \mathbb{N}$

$$\int_D f^p(\mathbf{x})\, \mathbb{E}(||W(\mathbf{x})||_k^n)\, \mathbb{E}(||\nabla X(\mathbf{x})||_{j,d}^{\ell})\, \mathbb{E}(||X(\mathbf{x})||_j^m)\, d\mathbf{x} < +\infty.$$

- **A$_2$**: For all $\mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where $F : \mathbb{R}^j \longrightarrow \mathbb{R}^j$ is a bijection of class C^1, such that $\forall\, \mathbf{z} \in \mathbb{R}^j$, the Jacobian of F at $\mathbf{z}$, $J_F(\mathbf{z})$ satisfies $J_F(\mathbf{z}) \neq 0$ and the function F^{-1} is continuous. The process $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is Gaussian of class C^1 on D in such a form that there exists a real $a > 0$, such that for almost surely $\mathbf{x} \in D$, $0 < a \leqslant \inf_{||z||_j=1} ||\mathbb{V}(Z(\mathbf{x})) \times z||_j$; the first order partial derivatives of its covariance Γ_Z are bounded almost surely on the diagonal contained in $D \times D$. Morever, for almost surely $\mathbf{x} \in D$, $W(\mathbf{x})$ is independent of the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$, and $\forall\, p \in \mathbb{N}$, $\forall\, n \in \mathbb{N}$, $\forall\, \ell \in \mathbb{N}$ and $\forall\, m \in \mathbb{N}$

$$\int_D f^p(\mathbf{x})\, \mathbb{E}(||W(\mathbf{x})||_k^n)\, \mathbb{E}(||\nabla Z(\mathbf{x})||_{j,d}^{\ell})\, \mathbb{E}(||Z(\mathbf{x})||_j^m)\, d\mathbf{x} < +\infty.$$

- **A$_3$**: For all $\mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^{j'}$ is Gaussian of class C^1 on D, with mean $m_Z(\cdot) = \mathbb{E}(Z(\cdot))$ bounded on D, and such that there exist reals a and b, $0 < a \leqslant b$ such that for almost surely $\mathbf{x} \in D$,

$$0 < a \leqslant \inf_{||z||_j=1} ||\mathbb{V}(Z(\mathbf{x})) \times z||_j \leqslant \sup_{||z||_j=1} ||\mathbb{V}(Z(\mathbf{x})) \times z||_j \leqslant b;$$

  the first order partial derivatives of its covariance Γ_Z are bounded almost surely on the diagonal contained in $D \times D$. Moreover, for almost surely $\mathbf{x} \in D$, $W(\mathbf{x})$ is independent of the vector

$(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ and we assume that this last vector has a density denoted by $\mathrm{p}_{Z(\mathbf{x}),\nabla Z(\mathbf{x})}(\cdot,\cdot)$. Finally $\forall\, p \in \mathbb{N}$, $\forall\, n \in \mathbb{N}$ and $\forall\, \ell \in \mathbb{N}$

$$\int_D f^p(\mathbf{x})\, \mathbb{E}(||W(\mathbf{x})||_k^n)\, \mathbb{E}(||\nabla Z(\mathbf{x})||_{j',d}^{\ell})\, d\mathbf{x} < +\infty. \tag{3.19}$$

The function F must satisfy assumption $(\mathbf{F})$ that is:

– $(\mathbf{F})$ $F : \mathbb{R}^{j'} \longrightarrow \mathbb{R}^j$ ($j < j'$) is of class C^2, furthermore defining $A_{j'} = \{1, 2, \ldots, j'\}$ there exists $\lambda = (\ell_1, \ell_2, \ldots, \ell_j) \in A_{j'}^j$, $\ell_1 < \ell_2 < \cdots, \ell_j$, such that $\forall \mathbf{z} \in \mathbb{R}^{j'}$,

$$J_F^{(\lambda)}(\mathbf{z}) = \det\left(\frac{\partial(F_1, \ldots, F_j)}{\partial(z_{\ell_1}, z_{\ell_2}, \ldots, z_{\ell_j})}(\mathbf{z})\right) \neq 0.$$

For simplicity reasons let us assume that $\lambda = (1, 2, \ldots, j)$ and let us denote $J_F(\mathbf{z})$ instead of $J_F^{(\lambda)}(\mathbf{z})$.
Moreover, $\forall\, \mathbf{v} \in \mathbb{R}^{j'-j}$, the function $F_\mathbf{v}$ defined by

$$\begin{aligned} F_\mathbf{v} : \mathbb{R}^j &\longrightarrow \mathbb{R}^j \\ \mathbf{u} &\longmapsto F_\mathbf{v}(\mathbf{u}) = F(\mathbf{u}, \mathbf{v}), \end{aligned}$$

is an invertible function whose inverse denoted $F_\mathbf{v}^{-1}$ is assumed to be a continuous function of the variable $\mathbf{u}$.
Also, $\forall \ell \in \mathbb{N}$ and $\forall \mu > 0$, the function T_ℓ defined by

$$\begin{aligned} T_\ell : \mathbb{R}^j &\longrightarrow \mathbb{R}^+ \\ \mathbf{u} &\longmapsto T_\ell(\mathbf{u}) \end{aligned}$$

$$= \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_\mathbf{z}^{-1}(\mathbf{u}), \mathbf{z})|} e^{-\mu\, ||\mathbf{z}||_{j'-j}^2}\, ||\nabla F(F_\mathbf{z}^{-1}(\mathbf{u}), \mathbf{z})||_{j,j'}^{\ell}\, d\mathbf{z}, \tag{3.20}$$

is continuous.

- $\mathbf{A_4}$: For almost surely $(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj}$ and for all $\mathbf{u} \in \mathbb{R}^j$, the density $\mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})$ of the joint distribution of $(Y(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x}))$ exists and is continuous in the variable $\mathbf{u}$. Moreover

$$\mathbf{u} \longmapsto \int_D \int_{\mathbb{R}\times\mathbb{R}^{dj}} |\mathbf{y}|||\dot{\mathbf{x}}||_{dj}^j \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x},$$

is continuous.

Remark 3.1.7 • It is interesting to notice that condition $\mathbf{A_1}$ (resp. $\mathbf{A_2}$, resp. $\mathbf{A_3}$) contains the case where the processes X and Y satisfy $\forall \mathbf{x} \in D$, $Y(\mathbf{x}) = G(\mathbf{x}, X(\mathbf{x}), \nabla X(\mathbf{x}))$ and also the case where $Y(\mathbf{x})$ is independent of $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ (resp. $Y(\mathbf{x})$ independent of $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$).

• Note also that condition $\mathbf{A_4}$ is satisfied for instance in the case where, $\forall u \in \mathbb{R}^j$, there exists a neighborhood V_u of u and a function h_u such that

$$\int_D \int_{\mathbb{R}\times\mathbb{R}^{dj}} |\mathbf{y}| ||\dot{\mathbf{x}}||_{dj}^j h_u(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x} < +\infty,$$

and such that for all $\mathbf{z} \in V_u$ and for almost surely

$$(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj},\ \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{z}, \dot{\mathbf{x}}) \leqslant h_u(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}).$$

In fact, these hypotheses are the ones which we need for applying the Lebesgue dominated convergence theorem that allows obtaining the continuity of the function

$$\mathbf{u} \longmapsto \int_D \int_{\mathbb{R}\times\mathbb{R}^{dj}} |\mathbf{y}| ||\dot{\mathbf{x}}||_{dj}^j \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x}.$$

We are now able to show a class of processes X and Y satisfying the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ through the following proposition.

Proposition 3.1.2 *If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ or if X and Y satisfy condition $\mathbf{A_4}$, then the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ hold.*

Proof of Proposition 3.1.2.

1. Let us first assume that the processes X and Y satisfy condition $\mathbf{A_1}$. Let us show that the hypotheses $\mathbf{H_3}$ and $\mathbf{H_5}$ are satisfied. Since X is Gaussian and that for almost surely $\mathbf{x} \in D$, $\inf_{||z||_j=1} ||\mathbb{V}(X(\mathbf{x})) \times z||_j \geqslant a > 0$, the distribution of vector $X(\mathbf{x})$ is not singular with density $\mathrm{p}_{X(\mathbf{x})}(\cdot)$. Moreover, $\mathbf{u} \to \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})$ is continuous and it is bounded by above, that is there exists a real M such that for almost surely $\mathbf{x} \in D$ and for all $\mathbf{u} \in \mathbb{R}^j$,

$$\mathrm{p}_{X(\mathbf{x})}(\mathbf{u}) \leqslant M. \tag{3.21}$$

The hypothesis $\mathbf{H_3}$ is satisfied.
Let us show that the hypothesis $\mathbf{H_5}$ holds.
Since for almost surely $\mathbf{x} \in D$,

$$Y(\mathbf{x}) = G(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})),$$

by using the hypotheses on G and since for all $\mathbf{A} \in \mathfrak{L}(\mathbb{R}^{\mathbf{d}}, \mathbb{R}^{\mathbf{j}})$, $H(\mathbf{A}) \leqslant \mathbf{C} \, ||\mathbf{A}||_{j,d}^{j}$, for all $\mathbf{u} \in \mathbb{R}^j$ we have

$$\mathbb{E}[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]$$
$$= \mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}],$$

where L is a continuous function of all its variables belonging to $D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and such that $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|L(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant Q(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}), \tag{3.22}$$

where Q is a polynomial with positive coefficients and $f : D \longrightarrow \mathbb{R}^+$ and $h : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ are continuous functions.
For almost surely $\mathbf{x} \in D$ fixed, let us consider the regression equations: for $\mathbf{s} \in D$

$$\begin{aligned} X(\mathbf{s}) &= \alpha(\mathbf{s})X(\mathbf{x}) + \xi(\mathbf{s}), \\ \nabla X(\mathbf{s}) &= \sum_{i=1}^{j} \nabla \alpha_i(\mathbf{s}) X_i(\mathbf{x}) + \nabla \xi(\mathbf{s}), \end{aligned} \tag{3.23}$$

where $(\xi(\mathbf{s}), \nabla \xi(\mathbf{s}))$ is a Gaussian vector independent of $X(\mathbf{x})$.
In particular, $\alpha(\mathbf{x}) = Id_j$.
A covariance computation gives

$$\alpha(\mathbf{s}) = \Gamma_X(\mathbf{s}, \mathbf{x}) \times \Gamma_X^{-1}(\mathbf{x}, \mathbf{x}),$$

where we recall that Γ_X stands for the covariance matrix of X.
Thus for all $i, m = 1, \ldots, j$ and $\ell = 1, \cdots, d$,

$$(\nabla \alpha_i(\mathbf{s}))_{\ell,m} = (\frac{\partial \Gamma_X}{\partial \mathbf{s}_\ell}(\mathbf{s}, \mathbf{x}) \times \Gamma_X^{-1}(\mathbf{x}, \mathbf{x}))_{mi}.$$

In particular for almost surely $\mathbf{x} \in D$, $i, m = 1, \ldots, j$ and $\ell = 1, \cdots, d$,

$$(\nabla \alpha_i(\mathbf{x}))_{\ell,m} = (\frac{\partial \Gamma_X}{\partial \mathbf{s}_\ell}(\mathbf{x},\mathbf{x}) \times \Gamma_X^{-1}(\mathbf{x},\mathbf{x}))_{mi}.$$

Since for almost surely $\mathbf{x} \in D$, $\inf_{||z||_j=1} ||\mathbb{V}(X(\mathbf{x})) \times z||_j \geqslant a > 0$ and the first order partial derivatives of the covariance Γ_X are bounded by above almost surely on the diagonal contained in $D \times D$, we get that there exists a real M such that for all $i = 1, \ldots, j$ and for almost surely $\mathbf{x} \in D$ we have

$$||\nabla \alpha_i(\mathbf{x})||_{j,d} \leqslant M. \tag{3.24}$$

For $\mathbf{u} \in \mathbb{R}^j$ let set

$$G_{X,L}(\mathbf{u}) = \int_D \mathbb{E}[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x}.$$

With the above notations we obtain then

$$G_{X,L}(\mathbf{u}) = \int_D \mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x}.$$

Since for almost surely $\mathbf{x} \in D$ the random variable $W(\mathbf{x})$ is independent of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$, using (3.23), this yields that

$$G_{X,L}(\mathbf{u}) =$$

$$\int_{D\times\Omega} L(\mathbf{x}, W(\mathbf{x})(\omega), \mathbf{u}, \sum_{i=1}^{j} \nabla \alpha_i(\mathbf{x})(\mathbf{u}_i - X_i(\mathbf{x})(\omega)) + \nabla X(\mathbf{x})(\omega))$$

$$\times\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, dP(\omega)\, d\mathbf{x}.$$

We have then eliminated the conditioning in the conditional expectation appearing into the integrant.
Now since for almost surely $\mathbf{x} \in D$, the function $\mathbf{u} \to \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})$ is continuous and since the function L is also continuous then for almost surely $(\omega, \mathbf{x}) \in \Omega \times D$, the function

$$\mathbf{u} \to$$

$$L(\mathbf{x}, W(\mathbf{x})(\omega), \mathbf{u}, \sum_{i=1}^{j} \nabla \alpha_i(\mathbf{x})(\mathbf{u}_i - X_i(\mathbf{x})(\omega)) + \nabla X(\mathbf{x})(\omega))\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u}),$$

is continuous.
Morever using the bounds (3.22) and (3.24), we get that for almost surely $(\omega, \mathbf{x}) \in \Omega \times D$,

$$|L(\mathbf{x}, W(\mathbf{x})(\omega), \mathbf{u}, \sum_{i=1}^{j} \nabla\alpha_i(\mathbf{x})(\mathbf{u}_i - X_i(\mathbf{x})(\omega)) + \nabla X(\mathbf{x})(\omega))| \, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})$$
$$\leqslant S(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, \ell(\mathbf{u}), ||X(\mathbf{x})(\omega)||_j, ||\nabla X(\mathbf{x})(\omega)||_{j,d}),$$

where S is also a polynomial with positive coefficients and $\ell : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ is a continuous function.
It is clear that for almost surely $(\omega, \mathbf{x}) \in \Omega \times D$, the function

$$\mathbf{u} \mapsto S(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, \ell(\mathbf{u}), ||X(\mathbf{x})(\omega)||_j, ||\nabla X(\mathbf{x})(\omega)||_{j,d}),$$

is continuous. Furthermore, we know that for $\forall\, p \in \mathbb{N}$, $\forall\, n \in \mathbb{N}$, $\forall\, \ell \in \mathbb{N}$ and for all $m \in \mathbb{N}$,

$$\int_D f^p(\mathbf{x}) \, \mathbb{E}(||W(\mathbf{x})||_k^n) \, \mathbb{E}(||\nabla X(\mathbf{x})||_{j,d}^{\ell}) \, \mathbb{E}(||X(\mathbf{x})||_j^m) \, d\mathbf{x} < +\infty.$$

Moreover recalling that

$$\int_{D\times\Omega} S(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, \ell(\mathbf{u}), ||X(\mathbf{x})(\omega)||_j,$$
$$||\nabla X(\mathbf{x})(\omega)||_{j,d}) dP(\omega) d\mathbf{x}$$
$$= \int_D \mathbb{E}(S(f(\mathbf{x}), ||W(\mathbf{x})||_k, \ell(\mathbf{u}), ||X(\mathbf{x})||_j, ||\nabla X(\mathbf{x}))||_{j,d})) \, d\mathbf{x},$$

and since the function $\mathbf{u} \mapsto \ell(\mathbf{u})$ is continuous we obtain that the function

$$\mathbf{u} \mapsto$$
$$\int_{D\times\Omega} S(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, \ell(\mathbf{u}), ||X(\mathbf{x})||_j, ||\nabla X(\mathbf{x})(\omega)||_{j,d}) dP(\omega) d\mathbf{x},$$

is continuous.
A weak application of the Lebesgue dominated convergence theorem allows to conclude that the hypothesis $\mathbf{H_5}$ holds true.
A similar proof can be made to show that the hypothesis $\mathbf{H_2}$ is also satisfied. This ends the first part of the proof.

2. Let assume now that the processes X and Y satisfy condition $\mathbf{A_2}$. Let us prove that hypothesis $\mathbf{H_3}$ is satisfied.
In the same form that in part 1) of the proof, since Z is Gaussian and given that for almost surely $\mathbf{x} \in D$,

$$\inf_{||z||_j=1} ||\mathbb{V}(Z(\mathbf{x})) \times z||_j \geqslant a > 0,$$

the distribution of the vector $Z(\mathbf{x})$ is non singular with density $\mathrm{p}_{Z(\mathbf{x})}(\cdot)$. The hypotheses on the function F entail that for almost surely $\mathbf{x} \in D$ the vector $X(\mathbf{x})$ has a density $\mathrm{p}_{X(\mathbf{x})}(\cdot)$ given by for all $\mathbf{u} \in \mathbb{R}^j$:

$$\mathrm{p}_{X(\mathbf{x})}(\mathbf{u}) = \frac{1}{|J_F(F^{-1}(\mathbf{u}))|}\,\mathrm{p}_{Z(\mathbf{x})}(F^{-1}(\mathbf{u})).$$

Let us show that the hypotheses $\mathbf{H_5}$ and $\mathbf{H_2}$ are satisfied.
By using the same notations of part 1), for almost surely $\mathbf{x} \in D$ and for all $\mathbf{u} \in \mathbb{R}^j$ we have

$$\begin{aligned}
&\mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\,\mathrm{p}_{X(\mathbf{x})}(\mathbf{u}) = \\
&\mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), F(Z(\mathbf{x})), \nabla F(Z(\mathbf{x})) \times \nabla Z(\mathbf{x}))|Z(\mathbf{x}) = F^{-1}(\mathbf{u})] \\
&\qquad\qquad \times \mathrm{p}_{Z(\mathbf{x})}(F^{-1}(\mathbf{u})) \times \frac{1}{|J_F(F^{-1}(\mathbf{u}))|} = \\
&\mathbb{E}[\widetilde{L}(\mathbf{x}, W(\mathbf{x}), Z(\mathbf{x}), \nabla Z(\mathbf{x}))|Z(\mathbf{x}) = F^{-1}(\mathbf{u})] \\
&\qquad\qquad \times \mathrm{p}_{Z(\mathbf{x})}(F^{-1}(\mathbf{u})) \times \frac{1}{|J_F(F^{-1}(\mathbf{u}))|},
\end{aligned}$$

where the function $\widetilde{L}$ is defined $\forall\,(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ by

$$\widetilde{L}(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) = L(\mathbf{x}, \mathbf{z}, F(\mathbf{u}), \nabla F(\mathbf{u}) \times \mathbf{A}).$$

It is clear, since F is C^1 that $\widetilde{L}$ has the properties of L, that is $\widetilde{L}$ is a continuous function of its variables in $D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and that

$$|\widetilde{L}(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant \widetilde{Q}(f(\mathbf{x}), ||\mathbf{z}||_k, \widetilde{h}(\mathbf{u}), ||\mathbf{A}||_{j,d}),$$

where $\widetilde{Q}$ is a polynomial with positive coefficients and
$\widetilde{h} : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ is a continuous function.
We have shown by using the notations of 1) that for all $\mathbf{u} \in \mathbb{R}^j$

$$G_{X,L}(\mathbf{u}) = G_{Z,\widetilde{L}}(F^{-1}(\mathbf{u})) \times \frac{1}{|J_F(F^{-1}(\mathbf{u}))|}.$$

This leads us to the case considered in 1) where the process X is replaced by the process Z. The continuity of the function $\mathbf{u} \to G_{X,L}(\mathbf{u})$ is a consequence of the one of $G_{Z,\widetilde{L}}$ and the fact that the function F^{-1} is continuous and F belongs to C^1. This ends the second part of the proof.

3. Let us assume that the processes X and Y satisfy the condition $\mathbf{A_3}$. We need to prove that conditions $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ hold.
Let us prove firstly that for almost surely $\mathbf{x} \in D$, the distribution of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ has a density $\mathrm{p}_{X(\mathbf{x});\nabla X(\mathbf{x})}(\cdot;\cdot)$, and let us compute this density.
First at all consider the following notations.
The matrix $s_{v,u} = (s_{ik})_{\substack{1\leqslant i\leqslant v\\ 1\leqslant k\leqslant u}} \in \mathfrak{L}(\mathbb{R}^u, \mathbb{R}^v)$, defined by its generic element s_{ik}, will be identified to the matrix $s_{v,u}$ with row vector $s_{(v,u)} \in \mathbb{R}^{vu}$, defined by

$$s_{(v,u)} = (s_{11}, s_{21}, \ldots, s_{v1}, s_{12}, s_{22}, \ldots, s_{v2}, \ldots, s_{1u}, s_{2u}, \ldots, s_{vu}).$$

Using this notation we can introduce the following function

$$K : \mathbb{R}^j \times \mathbb{R}^{j'-j} \times \mathbb{R}^{jd} \times \mathbb{R}^{(j'-j)d} \longrightarrow \mathbb{R}^j \times \mathbb{R}^{j'-j} \times \mathbb{R}^{jd} \times \mathbb{R}^{(j'-j)d}$$

$$\left(\mathbf{t} = \mathbf{t}_{(\mathbf{j}',\mathbf{1})} = \begin{pmatrix} \mathbf{t_{j,1}} \\ \mathbf{t_{j'-j,1}} \end{pmatrix}; \mathbf{s} = \mathbf{s}_{(\mathbf{j}',\mathbf{d})} = \begin{pmatrix} \mathbf{s_{j,d}} \\ \mathbf{s_{j'-j,d}} \end{pmatrix}\right)$$
$$\longmapsto \left((F(\mathbf{t}))_{(j,1)}; \mathbf{t}_{(\mathbf{j'-j,1})}; (\nabla F(\mathbf{t}) \times \mathbf{s})_{(j,d)}; \mathbf{s}_{(\mathbf{j'-j,d})}\right)$$

The jacobian J_K of this transformation satisfies: $\forall(\mathbf{t}, \mathbf{s}) \in \mathbb{R}^{j'} \times \mathbb{R}^{j'd}$:

$$J_K(\mathbf{t}, \mathbf{s}) = (J_F(\mathbf{t}))^{d+1} \neq 0,$$

by hypothesis.
Furthermore, since F belongs to C^2 then K belongs to C^1. Moreover, K is bijective having an inverse K^{-1} given by

$$K^{-1}: \mathbb{R}^j \times \mathbb{R}^{j'-j} \times \mathbb{R}^{jd} \times \mathbb{R}^{(j'-j)d} \longrightarrow \mathbb{R}^j \times \mathbb{R}^{j'-j} \times \mathbb{R}^{jd} \times \mathbb{R}^{(j'-j)d}$$

$$\left(\mathbf{t} = \begin{pmatrix} \mathbf{t_{j,1}} \\ \mathbf{t_{j'-j,1}} \end{pmatrix}; \mathbf{s} = \begin{pmatrix} \mathbf{s_{j,d}} \\ \mathbf{s_{j'-j,d}} \end{pmatrix}\right)$$

$$\longmapsto \left(F^{-1}_{\mathbf{t_{j'-j,1}}}(\mathbf{t_{j,1}}); \mathbf{t_{j'-j,1}}; \left[\nabla F(F^{-1}_{\mathbf{t_{j'-j,1}}}(\mathbf{t_{j,1}}); \mathbf{t_{j'-j,1}})\right]^{-1}_{jj}\right.$$

$$\left. \times \left(\mathbf{s_{j,d}} - \left[\nabla F(F^{-1}_{\mathbf{t_{j'-j,1}}}(\mathbf{t_{j,1}}); \mathbf{t_{j'-j,1}})\right]_{jj'-j} \times \mathbf{s_{j'-j,d}}\right); \mathbf{s_{j'-j,d}}\right),$$

where we have denoted, if $A \in \mathfrak{L}(\mathbb{R}^{j'}, \mathbb{R}^j)$, by $[A]_{jj}$ the matrix A for which we retain only the j first columns and by $[A]_{jj'-j}$ the matrix A for which we retain the $j'-j$ last columns.
For all $\mathbf{x} \in D$ since

$$X(\mathbf{x}) = F(Z(\mathbf{x})) \text{ and } \nabla X(\mathbf{x}) = \nabla F(Z(\mathbf{x})) \times \nabla Z(\mathbf{x}),$$

we have

$$K(Z(\mathbf{x}), \nabla Z(\mathbf{x})) = \left(X(\mathbf{x}); (Z(\mathbf{x}))_{j'-j,1}; \nabla X(\mathbf{x}); (\nabla Z(\mathbf{x}))_{j'-j,d}\right).$$

We deduce that if for almost surely $\mathbf{x} \in D$,
$\mathrm{p}_{X(\mathbf{x});(Z(\mathbf{x}))_{j'-j,1};\nabla X(\mathbf{x});(\nabla Z(\mathbf{x}))_{j'-j,d}}(\cdot;\cdot;\cdot;\cdot)$ denotes the density of the vector

$$\left(X(\mathbf{x}); (Z(\mathbf{x}))_{j'-j,1}; \nabla X(\mathbf{x}); (\nabla Z(\mathbf{x}))_{j'-j,d}\right),$$

and $\mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}(\cdot;\cdot)$ the one of $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ then

$$\forall(\mathbf{u}; \mathbf{z_{j'-j,1}}; \mathbf{s_{j,d}}; \mathbf{s_{j'-j,d}}) \in \mathbb{R}^j \times \mathbb{R}^{j'-j} \times \mathbb{R}^{jd} \times \mathbb{R}^{(j'-j)d},$$

we have

$$\mathrm{p}_{X(\mathbf{x});(Z(\mathbf{x}))_{j'-j,1};\nabla X(\mathbf{x});(\nabla Z(\mathbf{x}))_{j'-j,d}}(\mathbf{u}; \mathbf{z_{j'-j,1}}; \mathbf{s_{j,d}}; \mathbf{s_{j'-j,d}}) = \mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}(K^{-1}(\mathbf{u}; \mathbf{z_{j'-j,1}}; \mathbf{s_{j,d}}; \mathbf{s_{j'-j,d}})) \times \frac{1}{|J_F(F_{\mathbf{z_{j'-j,1}}}(\mathbf{u}); \mathbf{z_{j'-j,1}})|^{d+1}}.$$

Finally, we get the density of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ by integrating this last expression. Hence for almost surely $\mathbf{x} \in D$ and

$\forall(\mathbf{u};\mathbf{s_{j,d}}) \in \mathbb{R}^j \times \mathbb{R}^{jd}$ we have

$$\begin{aligned}
&\mathrm{p}_{X(\mathbf{x});\nabla X(\mathbf{x})}(\mathbf{u};\mathbf{s_{j,d}}) = \\
&\int_{\mathbb{R}^{j'-j}\times\mathbb{R}^{(j'-j)d}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} \times \mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}((F_{\mathbf{z}}^{-1}(\mathbf{u});\mathbf{z}); \\
&\qquad ([\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}^{-1} \times (\mathbf{s_{j,d}} - [\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj'-j} \\
&\qquad\qquad \times \mathbf{s_{j'-j,d}});\mathbf{s_{j'-j,d}}))d\mathbf{s_{j'-j,d}}\,d\mathbf{z} \quad (3.25)
\end{aligned}$$

Remark 3.1.8 It is important to point out that the results shown above could be obtained by using the coarea formula. We referred to the reader to Corollary 4.18 page 68 of [28]. However, we have preferred explicit computations in a way to obtain the exact expression of this density and also for introducing some notations useful in what follows.

Now for $\mathbf{u} \in \mathbb{R}^j$, set as in part 1)

$$\begin{aligned}
&G_{X,L}(\mathbf{u}) = \\
&\int_D \mathbb{E}[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\,\mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\,d\mathbf{x} \\
&\qquad = \int_D \mathbb{E}[L(\mathbf{x},W(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\,\mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\,d\mathbf{x}.
\end{aligned}$$

Since for almost surely $\mathbf{x} \in D$, $W(\mathbf{x})$ is independent of

$$(X(\mathbf{x}),\nabla X(\mathbf{x}))\,,$$

we get $\forall \mathbf{u} \in \mathbb{R}^j$,

$$\begin{aligned}
&G_{X,L}(\mathbf{u}) = \\
&\int_{D\times\mathbb{R}^{jd}} \mathbb{E}[L(\mathbf{x},W(\mathbf{x});\mathbf{u};\mathbf{s_{j,d}})]\mathrm{p}_{X(\mathbf{x});\nabla X(\mathbf{x})}(\mathbf{u};\mathbf{s_{j,d}})\,d\mathbf{s_{j,d}}\,d\mathbf{x} \\
&\qquad = \int_D \int_{\mathbb{R}^{j'-j}\times\mathbb{R}^{(j'-j)d}\times\mathbb{R}^{jd}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} \\
&\qquad \times \mathbb{E}[L(\mathbf{x},W(\mathbf{x});\mathbf{u};\mathbf{s_{j,d}})] \times \mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}((F_{\mathbf{z}}^{-1}(\mathbf{u});\mathbf{z}); \\
&([\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})_{jj}^{-1} \times (\mathbf{s_{j,d}} - [\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj'-j} \times \mathbf{s_{j'-j,d}}); \\
&\qquad\qquad \mathbf{s_{j'-j,d}}))d\mathbf{s_{j,d}}d\mathbf{s_{j'-j,d}}d\mathbf{z}d\mathbf{x}
\end{aligned}$$

In the last integral where the domain of integration is $\mathbb{R}^{jd}$, maintaining $(\mathbf{s_{j'-j,d}};\mathbf{z};\mathbf{x}) \in \mathbb{R}^{(j'-j)d} \times \mathbb{R}^{j'-j} \times D$ fixed, let us make the following change of variable

$$\mathbf{v_{j,d}} = [\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}^{-1} \times (\mathbf{s_{j,d}} - \left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{jj'-j} \times \mathbf{s_{j'-j,d}}),$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}^{jd}} \mathbb{E}[L(\mathbf{x},W(\mathbf{x});\mathbf{u};\mathbf{s_{j,d}})]\mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}((F_{\mathbf{z}}^{-1}(\mathbf{u});\mathbf{z});([\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}^{-1}\\
&\qquad \times(\mathbf{s_{j,d}} - \left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{jj'-j}\mathbf{s_{j'-j,d}});\mathbf{s_{j'-j,d}}))d\mathbf{s_{j,d}} =\\
&\int_{\mathbb{R}^{jd}} \mathbb{E}[L(\mathbf{x},W(\mathbf{x});\mathbf{u};[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj} \times \mathbf{v_{j,d}}\\
&\qquad + [\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj'-j} \times \mathbf{s_{j'-j,d}})]\\
&\quad \times \mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}((F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z});(\mathbf{v_{j,d}};\mathbf{s_{j'-j,d}})) \times |J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^d d\mathbf{v_{j,d}} =\\
&\int_{\mathbb{R}^{jd}} \mathbb{E}[L(\mathbf{x},W(\mathbf{x});\mathbf{u};\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\begin{pmatrix}\mathbf{v_{j,d}}\\ \mathbf{s_{j'-j,d}}\end{pmatrix})]\\
&\quad \times \mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}\left((F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z});\left(\mathbf{v_{j,d}};\mathbf{s_{j'-j,d}}\right)\right) \times |J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^d d\mathbf{v_{j,d}}.
\end{aligned}$$

Finally $\forall \mathbf{u} \in \mathbb{R}^j$,

$$\begin{aligned}
G_{X,L}(\mathbf{u}) &= \int_D \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|} \times\\
&\int_{\mathbb{R}^{(j'-j)d}\times\mathbb{R}^{jd}} \mathbb{E}[L\left(\mathbf{x},W(\mathbf{x});\mathbf{u};\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z}) \times \begin{pmatrix}\mathbf{v_{j,d}}\\ \mathbf{s_{j'-j,d}}\end{pmatrix}\right)]\times\\
&\mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}\left((F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z});\begin{pmatrix}\mathbf{v_{j,d}}\\ \mathbf{s_{j'-j,d}}\end{pmatrix}\right) d\mathbf{v_{j,d}}\, d\mathbf{s_{j'-j,d}}\, d\mathbf{z}\, d\mathbf{x}\\
&= \int_D \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|} \times\\
&\int_{\mathbb{R}^{j'd}} \mathbb{E}[L\left(\mathbf{x},W(\mathbf{x});\mathbf{u};\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z}) \times \mathbf{s_{j',d}}\right)]\times\\
&\mathrm{p}_{Z(\mathbf{x});\nabla Z(\mathbf{x})}((F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z});\mathbf{s_{j',d}})\, d\mathbf{s_{j',d}}\, d\mathbf{z}\, d\mathbf{x}. \quad (3.26)
\end{aligned}$$

A slight modification of this proof or the use of the coarea formula given in Corollary 4.18 page 68 of [28], show in first place, that for almost surely $\mathbf{x} \in D$ the vector $X(\mathbf{x})$ has a density $\mathrm{p}_{X(\mathbf{x})}(\cdot)$ given by: for all $\mathbf{u} \in \mathbb{R}^j$

$$\mathrm{p}_{X(\mathbf{x})}(\mathbf{u}) = \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} \times \mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})\, d\mathbf{z},$$

this implies the hypothesis $\mathbf{H_3}$. Moreover, this density is a continuous function of the variable $\mathbf{u}$.
Indeed, using the hypotheses on Z, we get the existence of a number $\lambda > 0$ such that for almost surely $\mathbf{x} \in D$ and for all $(\mathbf{z}, \mathbf{u}) \in \mathbb{R}^{j'-j} \times \mathbb{R}^j$,

$$\begin{aligned}
&\mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \\
&\qquad \leqslant \; \mathbf{C}\, e^{-2\lambda ||(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z}) - m_Z(\mathbf{x})||_{j'}^2} \leqslant \mathbf{C}\, e^{-\lambda ||(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})||_{j'}^2} \\
&\qquad \leqslant \; \mathbf{C}\, e^{-\lambda ||\mathbf{z}||_{j'-j}^2},
\end{aligned} \tag{3.27}$$

since the function $m_Z(\cdot)$ is bounded on D.
Furthermore, using the hypotheses satisfied by F and the process Z, for almost surely $\mathbf{z} \in \mathbb{R}^{j'-j}$ and $\mathbf{x} \in D$, the function

$$\mathbf{u} \mapsto \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} \times \mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})$$

is continuous and also the function $\mathbf{u} \mapsto \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|} e^{-\lambda ||\mathbf{z}||_{j'-j}^2}$.
By using that the function $\mathbf{u} \mapsto T_\ell(\mathbf{u})$ is continuous for $\ell = 0$ (see in the hypothesis $\mathbf{A_3}$, the equality (3.20)), an application of the Lebesgue dominated convergence theorem allows to assert that for almost surely $x \in D$, the function $\mathbf{u} \mapsto \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})$ is continuous. The hypothesis $\mathbf{H_3}$ holds.
Now coming back to the definition of $G_{X,L}(\mathbf{u})$ given by the equality (3.26), note that $G_{X,L}(\mathbf{u})$ can be written as

$$\begin{aligned}
G_{X,L}(\mathbf{u}) = \int_D \int_{\mathbb{R}^{j'-j}} & \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} \times \mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \\
& \mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), \mathbf{u}, \nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \nabla Z(\mathbf{x}))/Z(\mathbf{x}) \\
& \qquad = (F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})]\, d\mathbf{z}\, d\mathbf{x}.
\end{aligned}$$

In the same form that in part 1) of this proof, for all $\mathbf{s} \in D$ we do the regression of $Z(\mathbf{s})$ with respect to $Z(\mathbf{x})$ for almost surely $\mathbf{x} \in D$ so

$$Z(\mathbf{s}) = \alpha(\mathbf{s})Z(\mathbf{x}) + \xi(\mathbf{s}),$$
$$\nabla Z(\mathbf{s}) = \sum_{i=1}^{j'} \nabla \alpha_i(\mathbf{s}) Z_i(\mathbf{x}) + \nabla \xi(\mathbf{s}),$$

where $(\xi(\mathbf{s}), \nabla\xi(\mathbf{s}))$ is a Gaussian vector independent of $Z(\mathbf{x})$. Using the hypotheses on the process Z, we get as in the part 1) of this proof the following inequality: $\exists M \in \mathbb{R}$ such that for all $i = 1, \ldots, j'$ and for almost surely $\mathbf{x} \in D$ we have

$$||\nabla \alpha_i(\mathbf{x})||_{j',d} \leqslant M. \tag{3.28}$$

Moreover, for almost surely $\mathbf{x} \in D$ and using that $W(\mathbf{x})$ is independent of $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$, we get that for almost surely $\mathbf{x} \in D$,

$$G_{X,L}(\mathbf{u}) = \int_D \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} \times \mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \\ \mathbb{E}[L(\mathbf{x}, W(\mathbf{x}), \mathbf{u}, \nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \\ \left(\sum_{i=1}^{j'} \nabla \alpha_i(\mathbf{x}) [((F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}))_i - Z_i(\mathbf{x})] + \nabla Z(\mathbf{x}) \right))] \, d\mathbf{z} \, d\mathbf{x}.$$

As in part 1) we have turned non conditional the conditional expectation appearing into the integral and

$$G_{X,L}(\mathbf{u}) = \int_D \int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} \times \mathrm{p}_{Z(\mathbf{x})}(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \\ L(\mathbf{x}, W(\mathbf{x})(\omega), \mathbf{u}, \nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}) \times \\ (\sum_{i=1}^{j'} \nabla \alpha_i(\mathbf{x}) [((F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z}))_i - Z_i(\mathbf{x})(\omega)] + \nabla Z(\mathbf{x})(\omega))) dP(\omega) d\mathbf{z} d\mathbf{x} \\ = \int_D \int_{\mathbb{R}^{j'-j}} \int_\Omega f(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x}) \, dP(\omega) \, d\mathbf{z} \, d\mathbf{x}$$

By the hypotheses satisfied by Z and F and since L is a continuous function, we obtain for almost surely $(\omega, \mathbf{z}, \mathbf{x}) \in \Omega \times \mathbb{R}^{j'-j} \times D$,

that the function $\mathbf{u} \mapsto f(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x})$ is continuous. Let us bound now the expression $f(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x})$.
By using the bounds (3.22), (3.27) and (3.28) we get that for almost surely $(\omega, \mathbf{z}, \mathbf{x}) \in \Omega \times \mathbb{R}^{j'-j} \times D$,

$$
\begin{aligned}
f(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x})| & \\
\leqslant \quad & \frac{\mathbf{C}}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} e^{-\lambda ||(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})||_{j'}^2} \times R(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, h(\mathbf{u}), \\
& \qquad ||\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})||_{j,j'}, ||(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})||_{j'}, \\
& \qquad\qquad ||\nabla Z(\mathbf{x})(\omega)||_{j',d}, ||Z(\mathbf{x})(\omega)||_{j'}),
\end{aligned}
$$

where R is a polynomial with positive coefficients and $h : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ is a continuous function.
By using that $\forall n \in \mathbb{N}$, $\exists M_n > 0$ such that $\forall \mathbf{y} \in \mathbb{R}^{j'}$,

$$
e^{-\lambda/2||\mathbf{y}||_{j'}^2} \times ||\mathbf{y}||_{j'}^n \leqslant M_n,
$$

we get that for almost surely $(\omega, \mathbf{z}, \mathbf{x}) \in \Omega \times \mathbb{R}^{j'-j} \times D$,

$$
\begin{aligned}
|f(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x})| & \\
\leqslant \quad & \frac{\mathbf{C}}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|} e^{-\mu ||\mathbf{z}||_{j'-j}^2} \times S(f(\mathbf{x}), ||W(\mathbf{x})(\omega)||_k, h(\mathbf{u}), \\
& ||\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})||_{j,j'}, ||\nabla Z(\mathbf{x})(\omega)||_{j',d}, ||Z(\mathbf{x})(\omega)||_{j'}) \\
= \quad & g(\mathbf{u}, \omega, \mathbf{z}, \mathbf{x}),
\end{aligned}
$$

where S is again a polynomial with positive coefficients and $\mu = \frac{\lambda}{2} > 0$.
It is clear that g is a continuous function in the variable $\mathbf{u}$ for almost surely $(\omega, \mathbf{z}, \mathbf{x}) \in \Omega \times \mathbb{R}^{j'-j} \times D$.
In one hand, by using the hypothesis on Z and also the hypothesis (3.19) we have that $\forall\, p \in \mathbb{N}$, $\forall\, n \in \mathbb{N}$, $\forall\, \ell \in \mathbb{N}$ and $\forall m \in \mathbb{N}$,

$$
\begin{aligned}
& \int_D f^p(\mathbf{x})\, \mathbb{E}(||W(\mathbf{x})||_k^n)\, \mathbb{E}(||\nabla Z(\mathbf{x})||_{j',d}^{\ell})\, \mathbb{E}(||Z(\mathbf{x})||_{j'}^m)\, d\mathbf{x} \leqslant \\
& \qquad \mathbf{C} \int_D f^p(\mathbf{x})\, \mathbb{E}(||W(\mathbf{x})||_k^n)\, \mathbb{E}(||\nabla Z(\mathbf{x})||_{j',d}^{\ell})\; d\mathbf{x} < +\infty.
\end{aligned}
$$

Moreover, let us recall that

$$\int_D \int_{\mathbb{R}^{j'-j}} \int_\Omega g(\mathbf{u},\boldsymbol{\omega},\mathbf{z},\mathbf{x})\, dP(\boldsymbol{\omega})\, d\mathbf{z}\, d\mathbf{x} = \int_{\mathbb{R}^{j'-j}} \frac{\mathbf{C}}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|}\, e^{-\mu||\mathbf{z}||^2_{j'-j}} \times$$
$$\int_D \mathbb{E}[S(f(\mathbf{x}), ||W(\mathbf{x})||_k, h(\mathbf{u}), ||\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})||_{j,j'}, ||\nabla Z(\mathbf{x})||_{j',d}, ||Z(\mathbf{x})||_{j'})]\, d\mathbf{x}\; d\mathbf{z}$$

In the other hand, since for all $\ell \in \mathbb{N}$ the functions h and T_ℓ are continuous in $\mathbf{u}$, we obtain that the same is true for the function $\mathbf{u} \mapsto \int_D \int_{\mathbb{R}^{j'-j}} \int_\Omega g(\mathbf{u},\boldsymbol{\omega},\mathbf{z},\mathbf{x})\, dP(\boldsymbol{\omega})\, d\mathbf{z}\, d\mathbf{x}$.
By applying the convergence dominate theorem we get that hypothesis $\mathbf{H_5}$ holds true.
A similar proof allows us to show that hypothesis $\mathbf{H_2}$ is also satisfied. This ends the third part of the proof.

4. Let assume now that the processes X and Y satisfy the condition $\mathbf{A_4}$. Let us show that the hypothesis $\mathbf{H_5}$ holds true, since the hypothesis $\mathbf{H_3}$ is clearly satisfied.
For $\mathbf{u} \in \mathbb{R}^j$,

$$\int_D \mathbb{E}[Y(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x} = \int_D \int_{\mathbb{R}\times\mathbb{R}^{dj}} \mathbf{y}\, H(\dot{\mathbf{x}})\, \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y},\mathbf{u},\dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x}$$

Using the hypotheses on the density $\mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y},\mathbf{u},\dot{\mathbf{x}})$, we obtain that the function appearing into the integrant is a continuous function of the variable $\mathbf{u}$, for almost surely $(\mathbf{x},\mathbf{y},\dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj}$.
Besides, since for all $\mathbf{A} \in \mathfrak{L}(\mathbb{R}^{\mathbf{d}},\mathbb{R}^{\mathbf{j}})$, $H(\mathbf{A}) \leqslant \mathbf{C}\,||\mathbf{A}||^j_{j,d}$, readily we obtain the following bound for all $\mathbf{u} \in \mathbb{R}^j$, and almost surely $(\mathbf{x},\mathbf{y},\dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj}$,

$$|\mathbf{y}|\, H(\dot{\mathbf{x}})\, \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y},\mathbf{u},\dot{\mathbf{x}}) \leqslant \mathbf{C}\,|\mathbf{y}|\,||\dot{\mathbf{x}}||^j_{d,j}\, \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y},\mathbf{u},\dot{\mathbf{x}}) = g(\mathbf{x},\mathbf{y},\mathbf{u},\dot{\mathbf{x}}).$$

Now it is clear that for almost surely $(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj}$, the function $\mathbf{u} \longmapsto g(\mathbf{x}, \mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})$ remains continuous and also by hypothesis the function $\mathbf{u} \longmapsto \int_D \int_{\mathbb{R} \times \mathbb{R}^{dj}} g(\mathbf{x}, \mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x}$.
The dominated convergence theorem allows to get that the hypothesis $\mathbf{H_5}$ holds true. The same holds for the hypothesis $\mathbf{H_2}$.
This ends the proof of Proposition 3.1.2.

□

3.2 Rice's formula for all level

We have given before conditions for some classes of processes X and Y satisfying the hypotheses ($\mathbf{H_1}$, $\mathbf{H_4}$), or ($\mathbf{H_2}$, $\mathbf{H_3}$, $\mathbf{H_5}$).
Now we will provide a class of processes satisfying the hypotheses $\mathbf{H_i}$, $i = 1, 5$ simultaneously and will prove a proposition and then a theorem giving conditions on X and Y allowing the validity of the Rice's formula for all level. We must recall that Proposition 3.2.1 that follows was proved in 1985 by Cabaña [11]. Our proof is deeply inspired by this work.
The difficulty is in fact to exhibit a class of processes Y sufficiently large enough ensuring hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$, hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ being more simple to obtain. In this aim, that is to exhibit a class of processes verifying hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$, we required the only tool we provided before, that is using Proposition 3.1.1. Our objective is then to construct a class of processes Y satisfying assumption $\mathbf{A_0}$, that is such that Y is a continuous process for which there exists a $\lambda \in B_j$ in such a way that $\text{supp}(Y) \subset \Gamma(\lambda)$, $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$, $Y(\cdot)$ and $||\nabla X(\cdot)||_{j,d}$ are uniformly bounded on the support of Y.
These assumptions being very demanding, the idea consists in given a process Y verifying hypotheses $\mathbf{A_i}$, $i = 1, 4$, so that by Proposition 3.1.2 verifying hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$, approaching this last one for fixed $n \in \mathbb{N}^\star$, by a process $Y^{(n)}$, defined as $Y^{(n)} = \sum_{\lambda \in B_j} Y_\lambda^{(n)}$, where $Y_\lambda^{(n)}$ is still verifying hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ and above all assumption $\mathbf{A_0}$.
In that form, by Theorem 3.1.1 we will be able for fixed $n \in \mathbb{N}^\star$, to propose a Rice formula for processes X and $Y^{(n)}$ and for all level $\mathbf{y} \in \mathbb{R}^j$.

Then we will make n tends to infinity to get a Rice formula for X and Y.

In this aim let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\mathbf{C}^1(D, \mathbb{R}^j)$ where D is an open set of $\mathbb{R}^d$, and let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous processes. In the same form that in section 3.1.1, we define for fixed λ in B_j and $\mathbf{x} \in D^r_X$, $Y_\lambda(\mathbf{x}) = \eta_\lambda(\mathbf{x})Y(\mathbf{x})$, where $\eta_\lambda(\mathbf{t})$ was defined in (3.16).
For $n \in \mathbb{N}^*$, let us define the random variable $Y^{(n)}$ by

$$Y^{(n)}(\mathbf{x}) = \sum_{\lambda \in B_j} Y^{(n)}_\lambda(\mathbf{x}),$$

for $\mathbf{x} \in D$, where we define the random variable $Y^{(n)}_\lambda$ by

$$Y^{(n)}_\lambda(\mathbf{x}) = Y_\lambda(\mathbf{x}) f_n(\mathbf{x}) \Psi(Y(\mathbf{x})/n) \Psi(||\nabla X(\mathbf{x})||_{j,d}/n)$$
$$\Psi(1/(n\phi_\lambda(\mathbf{x}))\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})\, \mathbb{1}_{D^r_X}(\mathbf{x}),$$

where the function ϕ_λ was defined in (3.11), and Ψ is an even continuous function on $\mathbb{R}$, decreasing on $\mathbb{R}^+$ such that

$$\Psi(\mathbf{t}) = \begin{cases} 1, \ 0 \leqslant \mathbf{t} \leqslant 1 \\ 0, \ 2 \leqslant \mathbf{t} \end{cases}$$

and $(f_n)_{n \in \mathbb{N}^*}$ is the sequence of functions defined on $\mathbb{R}^d$ to $[0, 1]$ in the following manner

$$f_n(\mathbf{x}) = \frac{d(\mathbf{x}, D^{2n})}{d(\mathbf{x}, D^{2n}) + d(\mathbf{x}, D^{(n)})},$$

where the closed sets D^{2n} and $D^{(n)}$ are defined by

$$D^{2n} = \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, D^c) \leqslant \frac{1}{2n}\} \text{ and } D^{(n)} = \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, D^c) \geqslant \frac{1}{n}\}.$$

We will see later in the proof of the following Lemma 3.2.1 that the functions $(f_n)_{n \in \mathbb{N}^*}$ are well defined, continuous and such that the support of f_n/D is contained in D for each $n \in \mathbb{N}^*$. In Lemma 3.2.2 we will prove that $(f_n)_{n \in \mathbb{N}^*}$ is a sequence of nondecreasing functions tending to one when n goes to infinity.

Let us explain a little more the choice in the terms compounding the expression of $Y_\lambda^{(n)}$. The terms

- $Y_\lambda(\mathbf{x}) f_n(\mathbf{x}) \Psi(1/(n\phi_\lambda(\mathbf{x})) \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\,\mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})$ ensures that $Y_\lambda^{(n)}(\mathbf{x})$ will tend to $Y_\lambda(\mathbf{x})$ when n will be tend to infinity as $\mathbf{x} \in D_X^r$, as we will show using the fact that $f_n(\mathbf{x})$ tends to one when n goes to infinity. Then for $\mathbf{x} \in D_X^r$, $Y^{(n)}(\mathbf{x})$ will be tend to $\sum_{\lambda \in B_j} Y_\lambda(\mathbf{x}) = Y(\mathbf{x})$. Furthermore, this implies since for all $n \in \mathbb{N}^\star$ $\operatorname{supp}(f_n/D) \subset D$, that $\operatorname{supp}(Y_\lambda^{(n)}) \subset \Gamma(\lambda)$.

- $\Psi((1/n\phi_\lambda(\mathbf{x})) \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\,\mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})$ ensures that for all $n \in \mathbb{N}^\star$, $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$ is uniformly bounded on the support of $Y_\lambda^{(n)}$.

- $\Psi(Y(\mathbf{x})/n) f_n(\mathbf{x})$ ensures that $Y_\lambda^{(n)}$ is uniformly bounded on D since $f_n(\mathbf{x}) \leqslant 1$.

- $\Psi(||\nabla X(\mathbf{x})||_{j,d}/n)$ ensures that for all $n \in \mathbb{N}^\star$, $||\nabla X(\cdot)||_{j,d}$ is uniformly bounded on the support of $Y_\lambda^{(n)}$.

We can now establish the following lemmas.

Lemma 3.2.1 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\boldsymbol{C}^1(D, \mathbb{R}^j)$, where D is an open, convex and bounded set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ satisfying $\mathbb{E}(L_X(\cdot))^d < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process. Then in one hand, for $n \in \mathbb{N}^\star$, X and $Y^{(n)}$ are satisfying the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$. In the other hand, if Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{A_i}$, $i = 1, 2, 3$ or if X and Y satisfy the condition $\mathbf{A_4}$, then for all $n \in \mathbb{N}^\star$, X and $Y^{(n)}$ satisfy the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ and a fortiori the hypotheses $\mathbf{H_i}$, $i = 1, 5$.*

In this form we have provided a class of processes X and $Y^{(n)}$ satisfying simultaneously the hypotheses $\mathbf{H_i}$, $i = 1, 5$. Then by Theorem 3.1.1 we get that for all $n \in \mathbb{N}^\star$ and for all $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y^{(n)}(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right]$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\,\mathbb{E}\left[Y^{(n)}(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

The idea consists to make n tends to infinite. More precisely we can show the following lemma.

Lemma 3.2.2 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\boldsymbol{C}^1(D, \mathbb{R}^j)$, where D is an open, convex and bounded set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ satisfying $\mathbb{E}(L_X(\cdot))^d < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process. If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{A_i}$, $i = 1, 2, 3$ or if X and Y satisfy the condition $\mathbf{A_4}$, for all $\boldsymbol{y} \in \mathbb{R}^j$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y^{(n)}(\boldsymbol{x})d\sigma_{d-j}(\boldsymbol{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x})d\sigma_{d-j}(\boldsymbol{x})\right]$$

and

$$\lim_{n\to+\infty} \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\,\mathbb{E}\left[Y^{(n)}(\boldsymbol{x})H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\,\mathbb{E}\left[Y(\boldsymbol{x})H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Finally we can establish the following proposition.

Proposition 3.2.1 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\boldsymbol{C}^1(D, \mathbb{R}^j)$, where D is an open, convex and bounded set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^d < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process. If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{A_i}$, $i = 1, 2, 3$ or if X and Y satisfy $\mathbf{A_4}$, then for all $\boldsymbol{y} \in \mathbb{R}^j$ we have*

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x})d\sigma_{d-j}(\boldsymbol{x})\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\,\mathbb{E}\left[Y(\boldsymbol{x})H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Remark 3.2.1 In the same way as in Proposition 3.1.1 we can generalize this proposition by considering that D is an open and convex set not necessarily bounded, maintening the same hypotheses on process X. For Y, we will assume that it is defined on D_1 an open and bounded set included in D, adapting the hypotheses on Y to the open and bounded set D_1 instead of the open D and to X/D_1. The Rice's formula still holds for all level $\mathbf{y} \in \mathbb{R}^j$ and for X/D_1 and Y defined on D_1.

Proof of the Lemma 3.2.1. Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ be a random field belonging to $\mathbf{C}^1(D, \mathbb{R}^j)$ such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^d < +\infty$ and let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process.
Let us show that for all $n \in \mathbb{N}^\star$, the processes X and $Y^{(n)}$ satisfy the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$.
Let us begin considering for n fixed in $\mathbb{N}^\star$ and for λ fixed in B_j the process $Y_\lambda^{(n)}$. We will prove now that the process X and $Y_\lambda^{(n)}$ satisfy the condition $\mathbf{A_0}$ stated before the Proposition 3.1.1. In this form we will deduce by using this proposition that these processes will satisfy the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$.
Let us check that $Y_\lambda^{(n)}$ is continuous on D, that is a non trivial fact due to the presence of the indicator function of the set D_X^r into the definition of this function. Firstly let us remark that as the sets $(D^{2n})_{n\in\mathbb{N}^*}$ and $(D^{(n)})_{n\in\mathbb{N}^*}$ are closed, the functions $(f_n)_{n\in\mathbb{N}^*}$ are well defined and continuous on $\mathbb{R}^d$ and then on D. Let us now consider $\mathbf{x} \in (D_X^r)^{c_1}$ and a sequence $(\mathbf{x}_p)_{p\in\mathbb{N}^\star}$ of points belonging to D that converges to $\mathbf{x}$ when p tends to infinite. We have $Y_\lambda^{(n)}(\mathbf{x}) = 0$. Let assume that there exists a subsequence $(\mathbf{x}_{p_k})_{k\in\mathbb{N}^\star}$ of $(x_p)_{p\in\mathbb{N}^\star}$, such that $Y_\lambda^{(n)}(\mathbf{x}_{p_k}) \neq 0$, for all $k \in \mathbb{N}^\star$. In this case, necessarily $\phi_\lambda(\mathbf{x}_{p_k}) \geqslant \frac{1}{2n}$ for all $k \in \mathbb{N}^\star$, and since the function ϕ_λ is continuous on D, it holds that $\phi_\lambda(\mathbf{x}) \geqslant \frac{1}{2n}$. The property (3.13) implies that $\mathbf{x} \in \Gamma(\lambda) \subset D_X^r$, giving a contradiction. All the points except maybe a finite number, of the sequence $(\mathbf{x}_p)_{p\in\mathbb{N}^\star}$, are such that $Y_\lambda^{(n)}(\mathbf{x}_p) = 0$. The sequence $(Y_\lambda^{(n)}(\mathbf{x}_p))_{p\in\mathbb{N}^\star}$ converges then towards zero. Furthermore by using a reasoning similar to the precedent one we can prove that function $\mathbf{x} \to \Psi(1/(n\phi_\lambda(\mathbf{x}))\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})$ is continuous on D and then on D_X^r. Then the function $Y_\lambda^{(n)}$ is continuous on D_X^r that is an open set, yielding the continuity of $Y_\lambda^{(n)}$ on D.

Now let us prove that for all $n \in \mathbb{N}^\star$, the support of this function is contained in $\Gamma(\lambda)$, i.e. $\text{supp}(Y_\lambda^{(n)}) \subset \Gamma(\lambda)$.
To prove this inclusion, firstly we will prove that for all $n \in \mathbb{N}^*$, one has $\text{supp}(f_n/D) \subset D$.
Indeed, since for all $n \in \mathbb{N}^*$ the set D^{2n} is closed, we have

$$\text{supp}(f_n/D) = \overline{\{\mathbf{x} \in D, d(\mathbf{x}, D^c) > \frac{1}{2n}\}} \subset D,$$

the last inclusion is a consequence of the continuity of the distance function and of the fact that D is an open set.
Hence

$$\text{supp}(Y_\lambda^{(n)}) \subset \overline{\{\mathbf{x} \in D, \phi_\lambda(\mathbf{x}) \geqslant \frac{1}{2n}\}} \cap D \subset \{\mathbf{x} \in D, \phi_\lambda(\mathbf{x}) \geqslant \frac{1}{2n}\},$$

the last inclusion comes from the fact that function ϕ_λ is continuous on D. Finally the property (3.13) gives us $\text{supp}(Y_\lambda^{(n)}) \subset \Gamma(\lambda)$.
Let us see that $Y_\lambda^{(n)}$ is uniformly bounded on its support.
We only need to prove that $Y_\lambda^{(n)}$ is uniformly bounded on D. Let us consider $\mathbf{x} \in D$ such that $Y_\lambda^{(n)}(\mathbf{x}) \neq 0$. Then necessarily we have $|Y(\mathbf{x})| \leqslant 2n$. Since one has $f_n(\mathbf{x}) \leqslant 1$ and $\eta_\lambda(\mathbf{x})\mathbb{1}_{\{\mathbf{x} \in D_X^r\}} \leqslant 1$, $|Y_\lambda^{(n)}(\mathbf{x})| \leqslant |Y(\mathbf{x})|$.
Thus it holds that $|Y_\lambda^{(n)}(\mathbf{x})| \leqslant 2n$ and yields the result.
Let us show that $||(\nabla X(\cdot)/_{V_\lambda^\perp})^{-1}||_{d,j}$ is uniformly bounded on the support of $Y_\lambda^{(n)}$. We have seen that $\text{supp}(Y_\lambda^{(n)}) \subset \{\mathbf{x} \in D, \phi_\lambda(\mathbf{x}) \geqslant \frac{1}{2n}\}$. Then for $\mathbf{x} \in \text{supp}(Y_\lambda^{(n)})$, we have $||(\nabla X(\mathbf{x})/_{V_\lambda^\perp})^{-1}||_{d,j} \leqslant 2n$. Hence the result holds true.
Finally let us show that $||\nabla X(\cdot)||_{j,d}$ is uniformly bounded on the support of $Y_\lambda^{(n)}$. This follows from the following inclusion.

$$\text{supp}(Y_\lambda^{(n)}) \subset \overline{\{\mathbf{x} \in D, ||\nabla X(\mathbf{x})||_{j,d} \leqslant 2n\}} \cap D$$

$$\subset \{\mathbf{x} \in D, ||\nabla X(\mathbf{x})||_{j,d} \leqslant 2n\},$$

the last inclusion come from the fact that X belongs to C^1 on D.
Finally the processes X and $Y_\lambda^{(n)}$ satisfy the condition $\mathbf{A_0}$ and then the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$.

By using that $Y^{(n)} = \sum_{\lambda \in B_j} Y_\lambda^{(n)}$ and that $|Y^{(n)}| = \sum_{\lambda \in B_j} |Y_\lambda^{(n)}|$, it is clear that X and $Y^{(n)}$ satisfy also the hypotheses $\mathbf{H_1}$ and $\mathbf{H_4}$.
Let assume now that Y satisfy the condition (3.18) and that X and Y satisfy one of the conditions $\mathbf{A_i}$, $i = 1, 2, 3$. Let us prove then that X and $Y^{(n)}$ satisfy the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$.
For almost surely $\mathbf{x} \in D$, we have

$$Y(\mathbf{x}) = G(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})), \tag{3.29}$$

where G is a continuous function on $D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and such that $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant P(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}).$$

For all $n \in \mathbb{N}^\star$ and $\mathbf{x} \in D$

$$Y^{(n)}(\mathbf{x}) = \sum_{\lambda \in B_j} \eta_\lambda(\mathbf{x}) Y(\mathbf{x}) f_n(\mathbf{x}) \Psi(Y(\mathbf{x})/n) \Psi(||\nabla X(\mathbf{x})||_{j,d}/n)$$
$$\Psi(1/(n\phi_\lambda(\mathbf{x}))\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})\, \mathbb{1}_{D_X^r}(\mathbf{x}).$$

We deduce that for all $n \in \mathbb{N}^\star$ and almost surely for all $\mathbf{x} \in D$,

$$Y^{(n)}(\mathbf{x}) = M_n(\mathbf{x}, Y(\mathbf{x}), \nabla X(\mathbf{x})), \tag{3.30}$$

where for all $n \in \mathbb{N}^\star$, M_n is a continuous function defined on $D \times \mathbb{R} \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$. The proof of this last assertion can be made in a similar way that the one used for proving the continuity of $Y_\lambda^{(n)}$ on D. Moreover, $\forall\, (\mathbf{x}, \mathbf{y}, \mathbf{A}) \in D \times \mathbb{R} \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|M_n(\mathbf{x}, \mathbf{y}, \mathbf{A})| \leqslant \mathbf{C}\, |\mathbf{y}|.$$

By (3.29), we have for all $n \in \mathbb{N}^\star$ and for almost surely $\mathbf{x} \in D$,

$$\begin{aligned} Y^{(n)}(\mathbf{x}) &= M_n(\mathbf{x}, G(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})); \nabla X(\mathbf{x})) \\ &= G_n(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})), \end{aligned}$$

where for all $n \in \mathbb{N}^\star$ and $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$G_n(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) = M_n(\mathbf{x}, G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}); \mathbf{A}).$$

It is clear that G_n inherits the properties of G and M_n that is G_n is a continuous function on $D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and is such that

$$\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j),$$

$$\begin{array}{rcl} |G_n(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant \mathbf{C}\, |G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| & \leqslant & \mathbf{C}\, P(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}) \\ & = & Q(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}), \end{array}$$

where Q is as P, a polynomial with positive coefficients and

$$f : D \longrightarrow \mathbb{R}^+ \text{ and } h : \mathbb{R}^j \longrightarrow \mathbb{R}^+,$$

are continuous functions. Finally $Y^{(n)}$ satisfies the condition (3.18) and X and $Y^{(n)}$ satisfy one of the three conditions $\mathbf{A_1}$, $\mathbf{A_2}$ or $\mathbf{A_3}$. By using Proposition 3.1.2, we proved that the hypotheses $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$ hold for X and $Y^{(n)}$. By using the first part of this lemma we can conclude that the hypotheses $\mathbf{H_i}$, $i = 1, 5$, are satisfied by X and $Y^{(n)}$.
Let assume now that X and Y satisfy the condition $\mathbf{A_4}$. Let us prove that for all $n \in \mathbb{N}^\star$, X and $Y^{(n)}$ satisfy $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_5}$.
Since for almost surely $(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) \in D \times \mathbb{R} \times \mathbb{R}^{dj}$ and for all $\mathbf{u} \in \mathbb{R}^j$, the density $\mathrm{p}_{Y(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})$ of the joint distribution

$$(Y(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x}))$$

exists (and is continuous in $\mathbf{u}$) then for almost surely $\mathbf{x} \in D$ and for all $\mathbf{u} \in \mathbb{R}^j$, the density $\mathrm{p}_{X(\mathbf{x})}(\mathbf{u})$ of $X(\mathbf{x})$ exists and $\mathbf{H_3}$ holds true.
Now by using (3.30) we have for all $n \in \mathbb{N}^\star$ and for all $\mathbf{u} \in \mathbb{R}^j$,

$$\begin{array}{rcl} L_n(\mathbf{u}) & = & \displaystyle\int_D \mathbb{E}[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x} \\ & = & \displaystyle\int_D \mathbb{E}[M_n(\mathbf{x}, Y(\mathbf{x}), \nabla X(\mathbf{x})) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x} \\ & = & \displaystyle\int_D \mathbb{E}[L_n(\mathbf{x}, Y(\mathbf{x}), \nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{u}]\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{u})\, d\mathbf{x}, \end{array}$$

where L_n is a continuous function on $D \times \mathbb{R} \times \mathbb{R}^{dj}$ and since for all $\mathbf{A} \in \mathfrak{L}(\mathbb{R}^{\mathbf{d}}, \mathbb{R}^{\mathbf{j}})$, $H(\mathbf{A}) \leqslant \mathbf{C}\, ||\mathbf{A}||_{j,d}^j$, we have $\forall n \in \mathbb{N}^\star$ et $\forall\, (\mathbf{x}, \mathbf{y}, \mathbf{A}) \in D \times \mathbb{R} \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|L_n(\mathbf{x}, \mathbf{y}, \mathbf{A})| \leqslant \mathbf{C}\, |\mathbf{y}|\, ||\mathbf{A}||_{j,d}^j.$$

Finally for all $n \in \mathbb{N}^\star$ and for all $\mathbf{u} \in \mathbb{R}^j$,

$$L_n(\mathbf{u}) = \int_D \int_{\mathbb{R}\times\mathbb{R}^{dj}} L_n(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}})\, \mathrm{p}_{Y(\mathbf{x}),X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x}.$$

In the same form as in the proof of Proposition 3.1.2 (4), the dominated convergence theorem entails that for all $n \in \mathbb{N}^\star$, the function $\mathbf{u} \longmapsto L_n(\mathbf{u})$ is continuous. Then the hypothesis $\mathbf{H_5}$ holds true. Also one can get in the same manner the hypothesis $\mathbf{H_2}$. This ends the proof of this lemma. □

Proof of the Lemma 3.2.2. Let us prove first that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y^{(n)}(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]. \quad (3.31)$$

Recall that for all $n \in \mathbb{N}^\star$ and $\mathbf{x} \in D$ we have

$$Y^{(n)}(\mathbf{x}) = \sum_{\lambda\in B_j} \eta_\lambda(\mathbf{x}) Y(\mathbf{x}) f_n(\mathbf{x}) \Psi(Y(\mathbf{x})/n) \Psi(||\nabla X(\mathbf{x})||_{j,d}/n)$$
$$\Psi(1/(n\phi_\lambda(\mathbf{x}))\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\,\mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})\, \mathbb{1}_{D_X^r}(\mathbf{x}).$$

Notice that for all $\mathbf{y} \in \mathbb{R}^j$ and for all $\mathbf{x} \in D$,

- $\lim_{n\to+\infty} Y^{(n)}(\mathbf{x}) \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})}(\mathbf{x}) = Y(\mathbf{x})\, \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})}(\mathbf{x})$
- $|Y^{(n)}(\mathbf{x})| \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})}(\mathbf{x}) \leqslant |Y(\mathbf{x})|\, \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})}(\mathbf{x})$
- $Y \cdot \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})} \in L^1(d\sigma_{d-j} \otimes dP)$

Let us establish the first assertion, proving first that for $\mathbf{x} \in D$, $f_n(\mathbf{x})$ tends to one when n goes to infinity.
Consider $\mathbf{x} \in D$. Since D^c is closed, we have that $d(\mathbf{x}, D^c) > 0$, and there exists an integer $n_0 \in \mathbb{N}^*$ such that $d(\mathbf{x}, D^c) \geqslant \frac{1}{n_0}$. Thus for $n \geqslant n_0$, we have $d(\mathbf{x}, D^c) \geqslant \frac{1}{n}$, that implies for $n \geqslant n_0$, $\mathbf{x} \in D^{(n)}$. In consequence for all $n \geqslant n_0$, $d(\mathbf{x}, D^{(n)}) = 0$, then $f_n(\mathbf{x}) = 1$ for all $n \geqslant n_0$.
Finally the first assertion is a consequence of inclusion (3.17), that is for all $\lambda \in B_j$ one has $\mathrm{supp}(\eta_\lambda) \cap D_X^r \subset \Gamma(\lambda)$. Indeed this last inclusion implies that for all $\lambda \in B_j$, for all $\mathbf{x} \in D_X^r \cap \Gamma^{c_1}(\lambda)$, $\eta_\lambda(\mathbf{x}) = 0$ and then for all $\lambda \in B_j$, for all $\mathbf{x} \in D_X^r$, $\lim_{n\to+\infty} Y_\lambda^{(n)}(\mathbf{x}) = \eta_\lambda(\mathbf{x}) Y(\mathbf{x})$ so that for

all $\mathbf{x} \in D_X^r$, $\lim_{n\to+\infty} Y^{(n)}(\mathbf{x}) = (\sum_{\lambda\in B_j} \eta_\lambda(\mathbf{x}))Y(\mathbf{x}) = Y(\mathbf{x})$.
The last assertion, can be proven in the following fashion. By using Lemma 3.2.1, we know that X and $Y^{(n)}$ satisfy $\mathbf{H_i}$, $i = 1,5$ and in particular $\mathbf{H_1}$, $\mathbf{H_2}$ and $\mathbf{H_3}$. The Remark 3.1.1 yields that for all $n \in \mathbb{N}^\star$ and for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y^{(n)}(\mathbf{x})|\, d\sigma_{d-j}(\mathbf{x})\right]$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Remarking that the sets $(D^{2n})_{n\in\mathbb{N}^*}$ and $(D^{(n)})_{n\in\mathbb{N}^*}$ which define the sequence $(f_n)_{n\in\mathbb{N}^*}$ are respectively decreasing and nondecreasing, we obtain that the sequence $(f_n)_{n\in\mathbb{N}^*}$ is nondecreasing. Since the function Ψ is an even function on $\mathbb{R}$ and decreasing on $\mathbb{R}^+$, the sequence $(|Y^{(n)}|)_{n\in\mathbb{N}^*}$ is a nondecreasing one.
We can apply the Beppo-Levi theorem and we have for all $\mathbf{y} \in \mathbb{R}^j$

$$\lim_{n\uparrow+\infty} \uparrow \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right].$$

Similarly for all $\mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n\uparrow+\infty} \uparrow \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| \mathbb{1}_{D_X^r}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}$$
$$= \lim_{n\uparrow+\infty} \uparrow \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| \mathbb{1}_{D_X^r}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx},$$

the last equality comes from the fact that for all $\mathbf{x} \in D$ we have

$$\mathbb{1}_{D_X^r}(\mathbf{x}) H(\nabla X(\mathbf{x})) = H(\nabla X(\mathbf{x})).$$

We obtain then for all $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} < +\infty, \qquad (3.32)$$

since X and Y satisfy one of the four conditions $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ or $\mathbf{A_4}$ and by consequence of Proposition 3.1.2, satisfy the hypothesis $\mathbf{H_2}$.
We have shown that $Y \cdot \mathbb{1}_{\mathcal{C}_X^{D^r}(\mathbf{y})} \in L^1(d\sigma_{d-j} \otimes dP)$. Then by using the Lebesgue dominated convergence theorem we can deduce (3.31).
Let us show that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n \to +\infty} \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} \qquad (3.33)$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

In the same manner that above, let us notice that for all $\mathbf{y} \in \mathbb{R}^j$ and for almost surely $\mathbf{x} \in D$,

- $\lim_{n \to +\infty} Y^{(n)}(\mathbf{x})\, H(\nabla X(\mathbf{x}))\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{y}) = Y(\mathbf{x})\, H(\nabla X(\mathbf{x}))\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})$
- $|Y^{(n)}(\mathbf{x})|\, H(\nabla X(\mathbf{x}))\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{y}) \leqslant |Y(\mathbf{x})|\ H(\nabla X(\mathbf{x}))\, \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})$
- $\mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) \mathrm{p}_{X(\mathbf{x})}(\mathbf{y}) | X(\mathbf{x}) = \mathbf{y}\right] < +\infty,$

The finiteness of the last expression results from that of the second integral in (3.32).
The Lebesgue dominated convergence theorem allows to write for all $\mathbf{y} \in \mathbb{R}^j$ and for almost surely $\mathbf{x} \in D$,

- $$\lim_{n \to +\infty} \mathbb{E}\left[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{p}_{X(\mathbf{x})}(\mathbf{y}) = \mathbb{E}\left[Y(\mathbf{x})\, H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})$$

Furthermore, for all $\mathbf{y} \in \mathbb{R}^j$ and almost surely for all $\mathbf{x} \in D$,

$$\bullet \left|\mathbb{E}\left[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\right| \leqslant$$
$$\mathbb{E}\left[|Y(\mathbf{x})|\, H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{p}_{X(\mathbf{x})}(\mathbf{y}) \in L^1(D, d\mathbf{x}),$$

the last assertion comes from the fact that the second integral in (3.32) is finite.
The Lebesgue dominated convergence theorem allows to obtain (3.33). This ends the proof of Lemma 3.2.2 and this fact establishes the proof of Proposition 3.2.1. □

This proposition leads us to the next Theorem 3.2.1 and this one allows us to weaken the hypotheses $\mathbf{A_i}$, $i = 1, 2, 3$, in the following form.
More precisely, our goal is to avoid assuming the existence of uniform lower (or upper) bounds for the variance of process Z appearing in these hypotheses.
In the three first conditions $\mathbf{B_1}$, $\mathbf{B_2}$ et $\mathbf{B_3}$, we will assume that Y can be written as in formula (3.18). Let D an open set of $\mathbb{R}^d$.

- $\mathbf{B_1}$: Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a Gaussian random field belonging to C^1 on D, such that for all $\mathbf{x} \in D$, the vector $X(\mathbf{x})$ has a density. Moreover, for almost surely $\mathbf{x} \in D$, the field $W(\mathbf{x})$ is independent of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$, and $\forall\, n \in \mathbb{N}$,
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\, d\mathbf{x} < +\infty.$$
- $\mathbf{B_2}$: For all $\mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where $F : \mathbb{R}^j \longrightarrow \mathbb{R}^j$ is a bijection of class C^1, such that $\forall\, \mathbf{z} \in \mathbb{R}^j$, the Jacobian of F in $\mathbf{z}$, that is $J_F(\mathbf{z})$ satisfies $J_F(\mathbf{z}) \neq 0$ and the function F^{-1} is continuous.
Let $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a Gaussian process belonging to C^1 on D such that for all $\mathbf{x} \in D$, the vector $Z(\mathbf{x})$ has a density. Moreover, for almost surely $\mathbf{x} \in D$, $W(\mathbf{x})$ is independent of the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$, and $\forall\, n \in \mathbb{N}$,
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\ d\mathbf{x} < +\infty.$$
- $\mathbf{B_3}$: For all $\mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^{j'}$ is a Gaussian random field belonging to C^1 on D such that for all $\mathbf{x} \in D$, the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ has a density. Moreover, for almost surely $\mathbf{x} \in D$, $W(\mathbf{x})$ is independent of the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$. Finally, $\forall\, n \in \mathbb{N}$
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\ d\mathbf{x} < +\infty.$$

The function F verifies assumption $(\mathbf{F})$ given in condition $\mathbf{A_3}$.

- $\mathbf{B_4}$: Is the same condition $\mathbf{A_4}$.

Let us now state the hypothesis $\mathbf{H_6}$.

- $\mathbf{H_6}$: For all $\mathbf{y} \in \mathbb{R}^j$,

$$\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}[|Y(\mathbf{x})|H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}]\, d\mathbf{x} < +\infty.$$

We are now ready to state the following theorem.

Theorem 3.2.1 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ $(j \leqslant d)$ be a random field belonging to $\mathbf{C}^1(D, \mathbb{R}^j)$, where D is an open and bounded convex set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^d < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process.*
If Y satisfies condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{B_i}$, $i = 1, 2, 3$ and the hypothesis $\mathbf{H_6}$ or if X and Y satisfy the condition $\mathbf{B_4}$, then for all $\boldsymbol{y} \in \mathbb{R}^j$ we have

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x}) H(\nabla X(\boldsymbol{x}))|X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Remark 3.2.2 Under the same hypotheses as those of Theorem 3.2.1, eliminating the condition $\mathbb{E}(L_X(\cdot))^d < +\infty$ and the hypothesis **H6**, we get the following inequality for all $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$
$$\leqslant \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Remark 3.2.3 As in Proposition 3.2.1 we can generalize the theorem and the Remark 3.2.2 considering that D is an open and convex set non necessarily bounded but maintaining the same hypotheses about

the process X. For Y one assumes that it is defined on D_1 open and bounded set included in D, then we will adapt the hypotheses on Y to D_1 instead of the open set D and to X/D_1.
The Rice's formula will still hold for all level $\mathbf{y} \in \mathbb{R}^j$ and for X/D_1 and Y defined on D_1.

Proof of Theorem 3.2.1. Let us consider for all $n \in \mathbb{N}^\star$, the sets $D_n = \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, D^c) > \frac{1}{n}\}$. For all $n \in \mathbb{N}^\star$, D_n is an open set included in D. Considering now the restrictions X/D_n and Y/D_n. It is clear that if Y satisfies condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{B_i}$, $i = 1, 2, 3$ then for all $n \in \mathbb{N}^\star$, Y/D_n satisfies condition (3.18) and X/D_n and Y/D_n satisfy one of the three conditions $\mathbf{A_i}$, $i = 1, 2, 3$, where we have replaced the open set D by the open set D_n. In fact, it is enough for this to point out that for all $n \in \mathbb{N}^*$, $D_n \subseteq \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, D^c) \geqslant \frac{1}{n}\}$ which is a compact set contained in D.
The set D_n may be eventually not convex but it holds that $D_n \subset D$, and this last set is convex.
We apply the Remark 3.2.1 which follows the Proposition 3.2.1 to X/D_n and to Y/D_n (resp. $|Y|/D_n$). We get: $\forall \mathbf{y} \in \mathbb{R}^j$ and for all $n \in \mathbb{N}^*$,

$$\mathbb{E}\left[\int_{\mathcal{C}^{D^r}_{D_n,X}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_{D_n} \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx},$$

and also $\forall \mathbf{y} \in \mathbb{R}^j$ and for all $n \in \mathbb{N}^*$,

$$\mathbb{E}\left[\int_{\mathcal{C}^{D^r}_{D_n,X}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_{D_n} \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Noting since D is an open set of $\mathbb{R}^d$, $\lim_{n \uparrow +\infty} \uparrow D_n = D$, the Beppo Levi theorem allows to obtain, $\forall \mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} < +\infty,$$

by using the hypothesis $\mathbf{H_6}$.
We can apply the Lebesgue dominate convergence theorem to get , $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

If X and Y satisfy the condition $\mathbf{B_4}$ which is other than the condition $\mathbf{A_4}$, the above equality is trivial since already settled in the Proposition 3.2.1.
The proof of Theorem 3.2.1 is over. □

Proof of the Remark 3.2.2. In the same way as in the proof of the Lemma 3.2.1, let us define for all $n \in \mathbb{N}^\star$, the r.v. $Z^{(n)}$ by $Z^{(n)}(\mathbf{x}) = \sum\limits_{\lambda \in B_j} Z_\lambda^{(n)}(\mathbf{x})$ for $\mathbf{x} \in D$, where we have defined for λ fixed in B_j the r.v. $Z_\lambda^{(n)}$ by

$$Z_\lambda^{(n)}(\mathbf{x}) = Y_\lambda^{(n)}(\mathbf{x}) \Psi(L_X(\cdot)/n),$$

where we recall that we defined the r.v. $Y_\lambda^{(n)}$ by

$$Y_\lambda^{(n)}(\mathbf{x}) = Y_\lambda(\mathbf{x}) f_n(\mathbf{x}) \Psi(Y(\mathbf{x})/n) \Psi(||\nabla X(\mathbf{x})||_{j,d}/n)$$
$$\Psi(1/(n\phi_\lambda(\mathbf{x}))\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})>0\}} + 2\, \mathbb{1}_{\{\phi_\lambda(\mathbf{x})=0\}})\, \mathbb{1}_{D_X^r}(\mathbf{x}).$$

Let us note that we cannot work as in Lemma 3.2.1, since we are not be able of showing that $Z_\lambda^{(n)}$ verifies hypotheses $\mathbf{H_2}$ and $\mathbf{H_5}$. In fact we cannot apply the results of the Proposition 3.1.2, since we cannot verify that $Z_\lambda^{(n)}$ verifies assumptions $\mathbf{A_i}$, $i = 1, 4$. Indeed for $\mathbf{x} \in D$, $Z_\lambda^{(n)}(\mathbf{x})$ depends of all the trajectory of process X via the term $\Psi(L_X(\cdot)/n)$.
The processes X and $Z_\lambda^{(n)}$ satisfy the hypotheses of the Remark 3.1.5

and by consequence X and $Z_{\lambda}^{(n)}$ satisfy $\mathbf{H_1}$. As in the proof of Lemma 3.2.1, we deduce that X and $Z^{(n)}$ satisfy $\mathbf{H_1}$.
Furthermore, assuming that Y satisfy the condition (3.18) and that X and Y satisfy one of the three conditions $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ or that X and Y satisfy $\mathbf{A_4}$, the Proposition 3.1.2 allows us to deduce that $\mathbf{H_3}$ is satisfied.
By Proposition 2.2.1 we get that for almost surely $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Z^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx},$$

thus for almost surely $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] \leqslant \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Always according to the Proposition 3.1.2 the processes X and Y satisfy $\mathbf{H_2}$. Since X and $Z^{(n)}$ satisfy $\mathbf{H_1}$, we deduce that the right and left hand side terms of the last inequality are continuous as function of the variable $\mathbf{y}$. Then the inequality holds true for all $\mathbf{y} \in \mathbb{R}^j$.
In the same form as in the proof of the Lemma 3.2.2, by using the Beppo-Levi theorem we obtain for all $\mathbf{y} \in \mathbb{R}^j$

$$\lim_{n \uparrow +\infty} \uparrow \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right].$$

We have shown that for all $\mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] \leqslant \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx},$$

this ends the proof of this remark whenever Y satisfy the condition (3.18) and that X and Y satisfy one of the three conditions $\mathbf{A_i}$, for $i = 1, 3$

or that X and Y satisfy the condition $\mathbf{A_4}$ and then the condition $\mathbf{B_4}$.
In the case where Y satisfies the condition (3.18) and X and Y satisfy one of the three conditions $\mathbf{B_i}$, for $i = 1, 3$, in the same form as in the proof of Theorem 3.2.1, we apply for all $n \in \mathbb{N}^*$ the above inequality to X/D_n and Y/D_n that satisfy one of the three conditions $\mathbf{A_i}$, for $i = 1, 3$, making then n tends to infinity. The Beppo Levi theorem gives the looking for result without assuring that the right hand side term is finite because we do not assume the hypothesis $\mathbf{H_6}$. □

Our goal in this stage of these notes is to propose a Rice's formula that holds true for all level but without the hypothesis $\mathbb{E}(L_X(\cdot))^d < +\infty$ that was given in the Theorem 3.2.1. We will propose in the following a little better that the inequality appearing in the Remark 3.2.2. To do this we will replace in that theorem one of the conditions $\mathbf{B_i}$, $i = 1, 4$ by a condition $\mathbf{B_i^\star}$ slightly more strong. In the three first conditions $\mathbf{B_1^\star}$, $\mathbf{B_2^\star}$ and $\mathbf{B_3^\star}$, we will make the hypothesis that Y can be written under the form (3.18).
More precisely let D an open set of $\mathbb{R}^d$ and consider the following conditions:

- $\mathbf{B_1^\star}$: It is the condition $\mathbf{B_1}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the density of the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}))$ exists.

- $\mathbf{B_2^\star}$: It is the condition $\mathbf{B_2}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}))$ exists.

- $\mathbf{B_3^\star}$: It is the condition $\mathbf{B_3}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}))$ exists.

- $\mathbf{B_4^\star}$: It is the condition $\mathbf{B_4}$, plus the following hypotheses.

$$\mathbf{u} \longmapsto \int_D \int_{\mathbb{R} \times \mathbb{R}^{dj}} \mathbf{y}^2 ||\dot{\mathbf{x}}||_{dj}^j \mathrm{p}_{Y(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})}(\mathbf{y}, \mathbf{u}, \dot{\mathbf{x}})\, d\dot{\mathbf{x}}\, d\mathbf{y}\, d\mathbf{x},$$

  is continuous.

For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2) \in D \times D \times \mathbb{R}^{dj} \times \mathbb{R}^{dj}$ and for all $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^j \times \mathbb{R}^j$, the density

$$\mathrm{p}_{X(\mathbf{x_1}),X(\mathbf{x_2}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2})}(\mathbf{u}, \mathbf{v}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2),$$

of the vector$(X(\mathbf{x_1}), X(\mathbf{x_2}), \nabla X(\mathbf{x_1}), \nabla X(\mathbf{x_2}))$ exists.

Moreover, for all $\mathbf{y} \in \mathbb{R}^j$, the function

$$(\mathbf{u}, \mathbf{v}) \longmapsto$$
$$\int_{D\times D} \int_{\mathbb{R}^{dj}\times\mathbb{R}^{dj}} ||\dot{\mathbf{x}}_1||_{dj}^j ||\dot{\mathbf{x}}_2||_{dj}^j \mathrm{p}_{X(\mathbf{x}_1),X(\mathbf{x}_2),\nabla X(\mathbf{x}_1),\nabla X(\mathbf{x}_2)}(\mathbf{u}, \mathbf{v}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2)$$
$$d\dot{\mathbf{x}}_1\, d\dot{\mathbf{x}}_2\, d\mathbf{x}_1\, d\mathbf{x}_2,$$

is bounded in a neighborhood of $(\mathbf{y}, \mathbf{y})$.

Let us express now the hypothesis $\mathbf{H}_6^\star$.

- $\mathbf{H}_6^\star$: For all $\mathbf{y} \in \mathbb{R}^j$, the function

$$(\mathbf{u}, \mathbf{v}) \longmapsto \int_{D\times D} \mathrm{p}_{X(\mathbf{x_1}),X(\mathbf{x_2})}(\mathbf{u}, \mathbf{v}) \times$$
$$\mathbb{E}[H(\nabla X(\mathbf{x_1}))H(\nabla X(\mathbf{x_2}))|X(\mathbf{x_1}) = \mathbf{u}, X(\mathbf{x_2}) = \mathbf{v}]\, d\mathbf{x_1}\, d\mathbf{x_2},$$

  is a bounded function in a neighborhood of $(\mathbf{y}, \mathbf{y})$.

Finally we can state the following theorem.

Theorem 3.2.2 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\mathbf{C}^1(D, \mathbb{R}^j)$, where D is an open and convex bounded set of $\mathbb{R}^d$, such that for almost surely $\omega \in \Omega$, $\nabla X(\omega)$ is Lipschitz. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process. If Y satisfies (3.18) and if X and Y satisfy one of the three conditions $\mathbf{B}_\mathbf{i}^\star$, $i = 1, 2, 3$ and the hypothesis $\mathbf{H_6}$ and $\mathbf{H}_6^\star$ or if X and Ysatisfy the condition $\mathbf{B}_4^\star$, then for all $\boldsymbol{y} \in \mathbb{R}^j$ we have*

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x}) H(\nabla X(\boldsymbol{x})) | X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Remark 3.2.4 In the same form as in Theorem 3.2.1, we can generalize this theorem to the case when D is an open an convex set not necessarily bounded. It is enough to make the same hypotheses for the process X adapting those of Y now defined on the bounded set D_1 included in D to D_1 instead of the open set D and to X/D_1.
The Rice's formula will be still true for all level $\mathbf{y} \in \mathbb{R}^j$ and for X/D_1 and Y defined on D_1.

Proof of Theorem 3.2.2. For all $\mathbf{z} \in \mathbb{R}^j$, and with the same notations as in the proof of Lemma 3.2.1 and the Remark 3.2.2, we have for all $n \in \mathbb{N}^*$,

$$|\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] - \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})} |Y^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]| = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})} |Y^{(n)}(\mathbf{x})|(1 - \Psi(L_X(\cdot)/n))\, d\sigma_{d-j}(\mathbf{x})\right].$$

Let us begin assuming that if Y satisfies the condition (3.18) then X and Y satisfy one of the three conditions $\mathbf{A_i}$, $i = 1,2,3$, instead of the three conditions $\mathbf{B_i}$ that appear in the conditions $\mathbf{B_i^\star}$.
By Lemma 3.2.1, since X and Y satisfy one of the four hypotheses $\mathbf{A_i}$, $i = 1,4$, for all $n \in \mathbb{N}^*$, X and $Y^{(n)}$ satisfy the hypotheses $\mathbf{H_3}$ and $\mathbf{H_2}$ (also $\mathbf{H_5}$). By Proposition 2.2.1, we have then for almost surely $\mathbf{z} \in \mathbb{R}^j$ and for all $n \in \mathbb{N}^\star$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})} |Y^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{z})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{z}\right] \mathrm{dx},$$

and a similar formula holds true for the r.v. $Y^{(n)}$.

Let us denote for simplicity reasons $\left(\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta} f(\mathbf{x}) d\mathbf{x}\right)$ the following multiple integral

$$(\frac{1}{2\delta})^j \left(\int_{\mathbf{y}_1-\delta}^{\mathbf{y}_1+\delta} \cdots \int_{\mathbf{y}_j-\delta}^{\mathbf{y}_j+\delta} f(\mathbf{x}) d\mathbf{x}\right),$$

where $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_j)$ and f is a positive or integrable function defined over $\Pi_{i=1}^j [\mathbf{y}_j - \delta, \mathbf{y}_j + \delta]$ and taking real values.

With this notation and by using the Schwarz inequality we get for all $n \in \mathbb{N}^*$, for all $\mathbf{y} \in \mathbb{R}^j$ and for all $\delta > 0$:

$$\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\Bigg|\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})}|Z^{(n)}(\mathbf{x})|d\sigma_{d-j}(\mathbf{x})\right] - \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{z})\,\mathbb{E}\left[|Y^{(n)}(\mathbf{x})|H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{z}\right]\mathrm{dx}\Bigg|\,d\mathbf{z} \leqslant$$
$$(\mathbb{E}[(1-\Psi(L_X(\cdot)/n))^4])^{\frac{1}{4}} \times (\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\mathbb{E}[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})}(Y^{(n)}(\mathbf{x}))^2 d\sigma_{d-j}(\mathbf{x})]d\mathbf{z})^{\frac{1}{2}}$$
$$\times (\mathbb{E}[\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\sigma_{d-j}(C_X^{D^r}(\mathbf{z}))d\mathbf{z}]^2)^{\frac{1}{4}}. \quad (3.34)$$

Let us consider the second term in the product in the right hand side of the last inequality

To this end let us remark that if Y satisfies the condition (3.18), and if X and Y satisfy one of the three condition $\mathbf{A_i}$, $i = 1, 2, 3$, or if X and Y satisfy the condition $\mathbf{B_4^\star}$, it is easy to prove, as in Proposition 3.1.2 that X and Y^2 still satisfy the hypothesis $\mathbf{H_2}$ and $\mathbf{H_3}$.

By Proposition 2.2.1 and since for all $n \in \mathbb{N}^*$ and for all

$$\mathbf{x} \in D,\ |Y^{(n)}(\mathbf{x})| \leqslant |Y(\mathbf{x})|,$$

we get the following inequalities, $\forall\, n \in \mathbb{N}^*$, $\forall\, \mathbf{y} \in \mathbb{R}^j$,

$$\limsup_{\delta\to 0}\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})}(Y^{(n)}(\mathbf{x}))^2 d\sigma_{d-j}(\mathbf{x})\right]d\mathbf{z} \leqslant$$
$$\limsup_{\delta\to 0}\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{z})}Y^2(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right]d\mathbf{z} \leqslant$$
$$\limsup_{\delta\to 0}\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{z})\,\mathbb{E}\left[Y^2(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{z}\right]\mathrm{dx}\,d\mathbf{z}$$
$$\to_{\delta\to 0}\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\,\mathbb{E}\left[Y^2(\mathbf{x})H(\nabla X(\mathbf{x}))|X(\mathbf{x}) = \mathbf{y}\right]\mathrm{dx} < +\infty.$$

The last convergence comes from the fact that X and Y^2 satisfy the hypothesis $\mathbf{H_2}$. We will study now the third term in the product in the right hand side (3.34).

By Remark 2.1.1 following the Theorem 2.1.1 if we apply the coarea formula to the functions $G = X$ and $f = \mathbb{1}_{\{\Pi_{i=1}^{j}[\mathbf{y}_i-\delta;\mathbf{y}_i+\delta]\}} \geqslant 0$ and to the borel set $B = D$, for all $\mathbf{y} \in \mathbb{R}^j$ we obtain

$$\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta} \sigma_{d-j}(C_X^{D^r}(\mathbf{z}))\, d\mathbf{z} = \int_D \mathbb{1}_{\{X(\mathbf{x})\in\Pi_{i=1}^{j}[\mathbf{y}_i-\delta;\mathbf{y}_i+\delta]\}}\, H(\nabla X(\mathbf{x}))\, d\mathbf{x}.$$

If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{A_i}$, $i = 1,2,3$, instead of the three conditions $\mathbf{B_i}$ that appear in the conditions $\mathbf{B_i^\star}$, the additional hypothesis in condition $\mathbf{B_i^\star}$, $i = 1,2,3$, insures that for almost surely $(\mathbf{x_1},\mathbf{x_2}) \in D \times D$, the density of the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}))$ exists. It is enough for convincing itself to make the computation of the density as in the proof of Proposition 3.1.2 and this yields equality (3.25).
Furthermore, if X and Y satisfy the hypothesis $\mathbf{B_4^\star}$, it is clear that the density of the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}))$ exists for almost surely $(\mathbf{x_1},\mathbf{x_2}) \in D \times D$.
Finally for all $\mathbf{y} \in \mathbb{R}^j$, we get by the hypothesis $\mathbf{H_6^\star}$ or by using the forth condition appearing in $\mathbf{B_4^\star}$ and using the same conventions of notation as above, we get

$$\limsup_{\delta\to 0} \mathbb{E}\left[\frac{1}{2\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta} \sigma_{d-j}(C_X^{D^r}(\mathbf{z}))\, d\mathbf{z}\right]^2 =$$
$$\limsup_{\delta\to 0}(\frac{1}{2\delta})^2 \int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\int_{D\times D} \mathrm{p}_{X(\mathbf{x_1}),X(\mathbf{x_2})}(\mathbf{z_1},\mathbf{z_2})\times$$

$$\mathbb{E}[H(\nabla X(\mathbf{x_1}))H(\nabla X(\mathbf{x_2}))|X(\mathbf{x_1}) = \mathbf{z_1}, X(\mathbf{x_2}) = \mathbf{z_2}]d\mathbf{x_1}d\mathbf{x_2}d\mathbf{z_1}d\mathbf{z_2} \leqslant \mathbf{C}.$$

Before taking the limit when δ tends to zero in the inequality (3.34), let us observe that in the same form as in the proof of Remark 3.2.2, X and $Z^{(n)}$ satisfy the hypotheses $\mathbf{H_1}$ (and $\mathbf{H_4}$). Moreover, we have seem in the beginning of this proof that for all $n \in \mathbb{N}^*$, X and $Y^{(n)}$ satisfy the hypothesis $\mathbf{H_2}$ (and $\mathbf{H_5}$).
Thus taking the limit when δ tends to zero, it holds for all $n \in \mathbb{N}^*$ and

for all $\mathbf{y} \in \mathbb{R}^j$

$$\Bigg| \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] - \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} \Bigg| \leqslant \mathbf{C} \left(\mathbb{E}[(1-\Psi(L_X(\cdot)/n))^4]\right)^{\frac{1}{4}} \times \left(\int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y^2(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}\right)^{\frac{1}{2}}$$

The idea is now to take the limit when n tends towards infinity in the last inequality. First, let us observe that by using the Lebesgue dominated convergence theorem, $\lim_{n\to+\infty} \mathbb{E}[(1-\Psi(L_X(\cdot)/n))^4] = 0$.
Hence for all $\mathbf{y} \in \mathbb{R}^j$:

$$\limsup_{n\to+\infty} \Bigg| \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] - \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} \Bigg| = 0$$

Furthermore, by using the Beppo Levi theorem, $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n\uparrow+\infty} \uparrow \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y^{(n)}(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} < +\infty,$$

the fact that the last integral is finite provides from Proposition 3.1.2. But Beppo Levi theorem also entails that, $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n\uparrow+\infty} \uparrow \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Z^{(n)}(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right].$$

Then, $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right] < +\infty, \tag{3.35}$$

(and also for $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} |Y(\mathbf{x})| d\sigma_{d-j}(\mathbf{x})\right]$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[|Y(\mathbf{x})| H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}).$$

Now replacing $|Z^{(n)}|$ by $Z^{(n)}$ and $|Y^{(n)}|$ by $Y^{(n)}$, similarly as in the precedent proof, we get for all $\mathbf{y} \in \mathbb{R}^j$:

$$\limsup_{n\to+\infty} \left| \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Z^{(n)}(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] - \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} \right| = 0.$$

By (3.35), the Lebesgue dominated convergence theorem entails, for all $\mathbf{y} \in \mathbb{R}^j$

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Z^{(n)}(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right].$$

Also $\forall \mathbf{y} \in \mathbb{R}^j$,

$$\lim_{n\to+\infty} \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y^{(n)}(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx} = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

This ends the proof of the theorem in the case when either Y satisfies the condition (3.18), X and Y satisfy one of the three conditions $\mathbf{B_i^\star}$, $i = 1,2,3$, where we replaced in the condition $\mathbf{B_i^\star}$ the condition $\mathbf{B_i}$ by $\mathbf{A_i}$. If X and Y satisfy the condition $\mathbf{B_4^\star}$, the theorem holds true.
Let assume now that Y satisfies condition (3.18) and X and Y satisfy one of the three conditions $\mathbf{B_i^\star}$, $i = 1,2,3$. We will proceed as in the proof of Theorem 3.2.1. Let us consider for all $n \in \mathbb{N}^\star$, the sets $D_n = \{\mathbf{x} \in \mathbb{R}^d,\, d(\mathbf{x}, D^c) > \frac{1}{n}\}$.
For all $n \in \mathbb{N}^\star$, D_n is an open set contained in D. We consider the restrictions X/D_n and Y/D_n. It is clear that if Y satisfies the condition (3.18) and if X and Y satisfy one of the conditions $\mathbf{B_i^\star}$, $i = 1,2,3$, then

for all $n \in \mathbb{N}^\star$, Y/D_n satisfies the condition (3.18) and X/D_n and Y/D_n satisfy one of the conditions $\mathbf{B}_\mathbf{i}^\star$, $i = 1,2,3$ and also the hypothesis $\mathbf{H}_\mathbf{6}^\star$, where we have replaced the open set D by the open set D_n and replaced in $\mathbf{B}_\mathbf{i}^\star$ the condition $\mathbf{B}_\mathbf{i}$ by $\mathbf{A}_\mathbf{i}$.
We apply the theorem to X/D_n and to Y/D_n (resp. $|Y|/D_n$). We get then $\forall \mathbf{y} \in \mathbb{R}^j$ and for all $n \in \mathbb{N}^*$,

$$\mathbb{E}\left[\int_{\mathcal{C}^{D^r}_{D_n,X}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]$$

$$= \int_{D_n} \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx},$$

similarly replacing Y by $|Y|$.
The hypothesis $\mathbf{H_6}$ allows when n tends towards infinity to apply the Lebesgue dominated convergence theorem in the above equality.
Ending the proof of the theorem. □

3.3 General Rice formulas for all level

3.3.1 Preliminaries for the general Rice formula

The two following propositions proved by Azaïs & Wschebor [6, p. 178-179] will provide the arguments for obtaining a general Rice formula for a random field not necessarily regular and for all level $\mathbf{y} \in \mathbb{R}^j$.

Proposition 3.3.1 *Let $Z : \Omega \times W \subset \Omega \times \mathbb{R}^\ell \to \mathbb{R}^{m+k}$ ($m \in \mathbb{N}, k \in \mathbb{N}^\star$), be a random field C^1, W an open set of $\mathbb{R}^\ell$, J a compact subset of W whose Hausdorff dimension is less or equal to m and $\mathbf{z}_0 \in \mathbb{R}^{m+k}$ fixed. We assume that Z satisfies the following hypothesis: for all $\mathbf{t} \in J$, the random vector $Z(\mathbf{t})$ has a density $p_{Z(\mathbf{t})}(\mathbf{v})$ such that there exists a $\mathbf{C} > 0$, a neighborhood $V_{\mathbf{z}_0}$ of $\mathbf{z}_0$ satisfying that for all $\mathbf{t} \in J$ and for all $\mathbf{v} \in V_{\mathbf{z}_0}$, $p_{Z(\mathbf{t})}(\mathbf{v}) \leqslant \mathbf{C}$.*
Then almost surely there is not point $\mathbf{t} \in J$ such that $Z(\mathbf{t}) = \mathbf{z}_0$.

Proof of the Proposition 3.3.1. For T a borelian of $\mathbb{R}^\ell$ contained in W, let us denote

$$R_{\mathbf{z_0}}(T) = \{\omega \in \Omega : \exists\, \mathbf{x} \in T\,, Z(\mathbf{x})(\omega) = \mathbf{z}_0\}.$$

Let J a compact set contained in W whose Haussdorff dimension is less or equal to m. Since $k \in \mathbb{N}^\star$, the euclidian Haussdorff measure of J of dimension $m+k$ is zero, that is $H_{m+k}(J) = 0$ (c.f. definition in [28]). By definition of the euclidian pre-measure of Haussdorff of J that defines $H_{m+k}(J)$, that is $H^\delta_{m+k}(J)$, we have

$$H_{m+k}(J) = 0 = \lim_{\delta \to 0} H^\delta_{m+k}(J).$$

Consider $\varepsilon > 0$ and $\eta > 0$ fixed. There exists $\delta_\varepsilon > 0$ such that for all $\delta \leqslant \delta_\varepsilon$, there exists a numerable set I and $(r_i)_{i\in I}$, $0 < r_i \leqslant \delta$ for all $i \in I$ such that $J \subset \cup_{i\in I} B(\mathbf{x}_i, r_i)$ and $\sum\limits_{i\in I} r_i^{m+k} \leqslant \varepsilon$.
Moreover, since W is open in $\mathbb{R}^\ell$, for all $\mathbf{y} \in J \subset W$ there exists a $r_\mathbf{y} > 0$ such that $B(\mathbf{y}, 2r_\mathbf{y}) \subset \overline{B}(\mathbf{y}, 2r_\mathbf{y}) \subset W$.
Given that $J \subset \cup_{\mathbf{y}\in J} B(\mathbf{y}, r_\mathbf{y})$ and since J is compact, there exist a finite covering $(B(\mathbf{y}_j, r_{\mathbf{y}_j}))_{j=1,m}$ satisfying $J \subset \cup_{j=1}^m B(\mathbf{y}_j, r_{\mathbf{y}_j})$, $\mathbf{y}_j \in J$ for all $j = 1, m$. Consider $r = \inf_{i=1,m} r_{\mathbf{y}_j}$ and C the compact set defined as $C = \cup_{j=1}^m \overline{B}(\mathbf{y}_j, 2r_{\mathbf{y}_j}) \subset W$.
Let set $R_{\varepsilon,\eta} = inf(\delta_\varepsilon, r/2, \frac{\mu}{2\eta})$ where μ is the constant defining the neighborhood of $\mathbf{z}_0$, where the density of Z is bounded. This last neighborhood satisfies that for all $\mathbf{t} \in J$ the random vector $Z(\mathbf{t})$ has a density $p_{Z(\mathbf{t})}(\mathbf{v})$ satisfying $p_{Z(\mathbf{t})}(\mathbf{v}) \leqslant \mathbf{C}$, for $\mathbf{v}$ such that $||\mathbf{v} - \mathbf{z}_0||_{m+k} \leqslant \mu$.
Thus there exists a numerable set I and $(r_i)_{i\in I}$, $0 < r_i \leqslant R_{\varepsilon,\eta}$ for all $i \in I$ satisfying $J \subset \cup_{i\in I} B(\mathbf{x}_i, r_i)$ and $\sum\limits_{i\in I} r_i^{m+k} \leqslant \varepsilon$. We have

$$P(R_{\mathbf{z_0}}(J)) \leqslant P(\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| > \eta)$$

$$+ \sum_{i\in I} P\left(\left\{\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| \leqslant \eta\right\} \cap R_{\mathbf{z}_0}(B(\mathbf{x}_i, r_i) \cap J)\right).$$

Set i fixed in I. If $B(\mathbf{x}_i, r_i) \cap J = \emptyset$, then $R_{\mathbf{z}_0}(B(\mathbf{x}_i, r_i) \cap J) = \emptyset$ and $P\left(\left\{\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| \leqslant \eta\right\} \cap R_{\mathbf{z}_0}(B(\mathbf{x}_i, r_i) \cap J)\right) = 0$. If $B(\mathbf{x}_i, r_i) \cap J \neq \emptyset$, let fix $\mathbf{z} \in B(\mathbf{x}_i, r_i) \cap J$.
For $\omega \in \left\{\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| \leqslant \eta\right\} \cap R_{\mathbf{z}_0}(B(\mathbf{x}_i, r_i) \cap J)$ there exists $\mathbf{x} \in B(\mathbf{x}_i, r_i) \cap J$ such that $Z(\mathbf{x})(\omega) = \mathbf{z}_0$.
Let us remark that there exists $j = 1, m$ such that $\mathbf{x}$ and $\mathbf{z}$ belong to the ball $B(\mathbf{y}_j, 2r_{\mathbf{y}_j})$, this entails that for all $\lambda \in [0, 1]$, $\lambda\mathbf{x} + (1-\lambda)\mathbf{z} \in C$.

Indeed, since $\mathbf{x} \in J$ there exists a $j = 1, m$, such that $\mathbf{x} \in B(\mathbf{y}_j, r_{\mathbf{y}_j})$. We have the following inequalities

$$\begin{aligned} ||\mathbf{z} - \mathbf{y}_j||_\ell &\leqslant ||\mathbf{z} - \mathbf{x}_i||_\ell + ||\mathbf{x}_i - \mathbf{x}||_\ell + ||\mathbf{x} - \mathbf{y}_j||_\ell \\ &\leqslant 2r_i + r_{\mathbf{y}_j} \leqslant 2R_{\varepsilon,\eta} + r_{\mathbf{y}_j} \leqslant r + r_{\mathbf{y}_j} \leqslant 2r_{\mathbf{y}_j}. \end{aligned}$$

Furthermore, since Z belongs to C^1 over W and then over $B(\mathbf{y}_j, 2r_{\mathbf{y}_j})$ which is an open convex set we have then

$$\begin{aligned} Z(\mathbf{z})(\omega) - Z(\mathbf{x})(\omega) &= Z(\mathbf{z})(\omega) - \mathbf{z}_0 \\ &= \left[\int_0^1 \nabla Z(\lambda \mathbf{x} + (1-\lambda)\mathbf{z})(\omega)\, d\lambda\right] (\mathbf{z} - \mathbf{x}), \end{aligned}$$

in consequence as for all $\lambda \in [0,1]$ we have $\lambda \mathbf{x} + (1-\lambda)\mathbf{z} \in C$ then

$$||Z(\mathbf{z})(\omega) - \mathbf{z}_0||_{m+k} \leqslant \eta\, ||\mathbf{z} - \mathbf{x}||_\ell \leqslant 2\eta r_i \leqslant 2\eta R_{\varepsilon,\eta} \leqslant \mu$$

Hence

$$\begin{aligned} &P\left(\left\{\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| \leqslant \eta\right\} \cap R_{\mathbf{z}_0}(B(\mathbf{x}_i, r_i) \cap J)\right) \\ &\qquad \leqslant P(\omega, ||Z(\mathbf{z})(\omega) - \mathbf{z}_0||_{m+k} \leqslant 2\eta r_i) \\ &\qquad = \int_{\mathbb{R}^{m+k}} \mathbb{1}_{\{||\mathbf{v}-\mathbf{z}_0||_{m+k} \leqslant 2\eta r_i\}}\, p_{Z(\mathbf{z})}(\mathbf{v})\, d\mathbf{v} \leqslant \mathbf{C}\, D_{m,k}\, (\eta r_i)^{m+k}. \end{aligned}$$

Finally we have shown that $\forall \varepsilon > 0$, $\forall \eta > 0$,

$$\begin{aligned} P(R_{\mathbf{z}_0}(J)) &\leqslant P(\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| > \eta) + \mathbf{C}\, D_{m,k}\, \eta^{m+k} \sum_{i\in I} r_i^{m+k} \\ &\leqslant P(\sup_{\mathbf{t}\in C} ||\nabla Z(\mathbf{t})|| > \eta) + \mathbf{C}\, D_{\ell,m,k}\, \eta^{m+k}\, \varepsilon \end{aligned}$$

By taking limits when ε tends to zero then when η tends to infinity, in this order, we get $P(R_{\mathbf{z}_0}(J)) = 0$. □

We are now able of stating the second proposition.

Proposition 3.3.2 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\mathbf{C}^2(D, \mathbb{R}^j)$, where D is an open set of $\mathbb{R}^d$ and let D_0 be a compact of $\mathbb{R}^d$ contained in D. Let $\mathbf{y} \in \mathbb{R}^j$ fixed. We assume that X satisfies the following assumption* $(\mathbf{S})$*:*

- $(\mathbf{S})$ *For all* $(\mathbf{x}, \lambda) \in D_0 \times S^{j-1}$ *the random vector*

$$(X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})),$$

has a density $p_{X(\mathbf{x}),\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w})$*, such that there exists a constant* $\mathbf{C} > 0$*, a neighborhood* $V_{\mathbf{y}}$ *of* $\mathbf{y}$ *and a neighborhood* $V_{\vec{0}_d}$ *of* $\vec{0}_{\mathbb{R}^d}$*, such that for all* $\mathbf{x} \in D_0$ *and for all* $\lambda \in S^{j-1}$*, for all* $\mathbf{u} \in V_{\mathbf{y}}$ *and for all* $\mathbf{w} \in V_{\vec{0}_d}$*,* $p_{X(\mathbf{x}),\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w}) \leqslant \mathbf{C}$.

Then

$$\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D_0,\ X(\mathbf{x})(\omega) = \mathbf{y}\,, \textit{rank}\, \nabla X(\mathbf{x})(\omega) < j\} = 0.$$

Proof of the Proposition 3.3.2. Let us define the random field Z by $Z : \Omega \times D \times \mathbb{R}^j \subset \Omega \times \mathbb{R}^d \times \mathbb{R}^j \to \mathbb{R}^j \times \mathbb{R}^d$ such that

$$Z(\mathbf{x}, \lambda) = (X(\mathbf{x}),\ \lambda \cdot \nabla X(\mathbf{x})).$$

Let us consider the set $W = D \times \mathbb{R}^j$ that is an open set of $\mathbb{R}^\ell$, where $\ell = d + j$. The field Z is C^1 on W since X is C^2 on D, and takes its values in $\mathbb{R}^{m+k}$ where $m = j + d - 1$ and $k = 1$.
Considering $J = D_0 \times S^{j-1}$ a compact contained in W. Its Haussdorff measure is less or equal to m. Set $\mathbf{z}_0 = (\mathbf{y}, \vec{0}_{\mathbb{R}^d}) \in \mathbb{R}^{m+k}$ fixed.
Since for all $(\mathbf{x}, \lambda) \in D_0 \times S^{j-1}$ the random vector $(X(\mathbf{x}),\ \lambda \cdot \nabla X(\mathbf{x}))$ has a bounded density $p_{X(\mathbf{x}),\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w})$, for $\mathbf{u}$ in a neighborhood of $\mathbf{y}$ and $\mathbf{w}$ in a neighborhood of $\vec{0}_{\mathbb{R}^d}$ then for all $\mathbf{t} \in J$ the random vector $Z(\mathbf{t})$ has a density $p_{Z(\mathbf{t})}(\mathbf{v})$ satisfying $p_{Z(\mathbf{t})}(\mathbf{v}) \leqslant \mathbf{C}$, for $\mathbf{v}$ in a neighborhood of $\mathbf{z}_0$.
By the Proposition 3.3.1,

$$\mathbb{P}\{\omega \in \Omega : \exists (\mathbf{x}, \lambda) \in D_0 \times S^{j-1}, (X(\mathbf{x})(\omega), \lambda \cdot \nabla X(\mathbf{x})(\omega)) = (\mathbf{y}, \vec{0}_{\mathbb{R}^d})\} = 0$$

then

$$\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D_0,\ X(\mathbf{x})(\omega) = \mathbf{y}\,,\ \text{rank}\ \nabla X(\mathbf{x})(\omega) < j\} = 0.$$

This ends the proof of the proposition. □

We have now all the ingredients to prove the general Rice formula for all level.

3.3.2 The general Rice formula

In this section the Theorem 3.3.1 provides a general Rice formula for all level. Let us point out that Theorem 6.10 of [6] gives the same result as that obtained in that one. However, in the aforementioned Theorem 6.10 the proofs are only sketched.
The proof of Theorem 3.3.1 will be based in the proof of the Theorem 3.2.1. Therefore, its proof will need more general conditions than those denoted by $\mathbf{B_i}$, $i = 1,4$, that appear in this last theorem. Thus let us state the following conditions $\mathbf{C_i}$, $i = 1,4$ with the same precedent convention. That is for the three first conditions $\mathbf{C_i}$, $i = 1,3$ the process Y will be expressed using (3.18).
In what follows we have the conditions.

- $\mathbf{C_1}$: It is the condition $\mathbf{B_1}$ plus the following hypothesis. For all $\mathbf{x} \in D$ the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ has a density.
- $\mathbf{C_2}$: It is the condition $\mathbf{B_2}$ plus the following hypothesis. For all $\mathbf{x} \in D$, the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ has a density.
- $\mathbf{C_3}$: It is the condition $\mathbf{B_3}$ plus the following hypothesis. The function F would be verified assumption $(\mathbf{FF})$ that is:
  - $(\mathbf{FF})$ For all $\mathbf{y} \in \mathbb{R}^j$ there exists $\mathbf{C} > 0$, there exists a neighborhood $V_{\mathbf{y}}$ of $\mathbf{y}$ such that for all $\mu > 0$ and for all $\mathbf{u} \in V_{\mathbf{y}}$ we have

$$\int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})|^{d+1}} e^{-\mu||\mathbf{z}||^2_{j'-j}} ||\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}), \mathbf{z})||_{jj'}^{(j-1)d} d\mathbf{z}$$
$$\leqslant \mathbf{C}.$$

- $\mathbf{C_4}$: It is the condition $\mathbf{B_4}$ plus the following hypothesis. The process X verifies the assumption $(\mathbf{S})$.

Let us state the general Rice formula for all level.

Theorem 3.3.1 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ $(j \leqslant d)$ be a random field belonging to $\mathbf{C}^2(D, \mathbb{R}^j)$, where D is a bounded convex open set of $\mathbb{R}^d$. We assume that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz of Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^d < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a*

continuous process.
If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{C_i}$, $i = 1, 2, 3$ and the hypothesis $\mathbf{H_6}$ or if X and Y satisfy hypothesis $\mathbf{C_4}$ then for all $\boldsymbol{y} \in \mathbb{R}^j$ we have

$$\mathbb{E}\left[\int_{\mathcal{C}_X(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right] = \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x}) H(\nabla X(\boldsymbol{x})) | X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Remark 3.3.1 We can replace the hypothesis "for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ having a d-order moment", by the hypothesis

$$\text{“}\mathbb{E}(\sup_{\mathbf{x} \in D} ||\nabla^2(X(\mathbf{x}))||_{j,d}^{(s)})^d < +\infty\text{”}.$$

Indeed if X is C^2 on D, the Taylor formula on D convex and open set allows concluding that almost surely

$$L_X = \sup_{\mathbf{x} \in D} ||\nabla^2(X(\mathbf{x}))||_{j,d}^{(s)} < +\infty,$$

then almost surely ∇X is Lipschitz with Lipschitz constant L_X.

Remark 3.3.2 We can generalize this theorem and also the remark to the case where D is an open set no necessarily bounded. It is enough to conserve the same hypotheses for X adapting those of Y that is defined on the open bounded set D_1 included in D to D_1 instead of the open set D and to X/D_1.
The Rice formula holds true for all $\mathbf{y} \in \mathbb{R}^j$ and for X/D_1 and Y defined on D_1.

Proof of the Theorem 3.3.1. By the Theorem 3.2.1, we already know that $\forall \mathbf{y} \in \mathbb{R}^j$

$$\mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]$$
$$= \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{dx}.$$

Let us verify that assumption $(\mathbf{S})$ holds. That is let us prove that if D_0 is a compact set contained in D and if X and Y satisfy one of the

conditions $\mathbf{C_i}$, $i = 1, 3$ then for all $(\mathbf{x}, \lambda) \in D_0 \times S^{j-1}$ the random vector $(X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x}))$ has a density $p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w})$ such that for all $\mathbf{y} \in \mathbb{R}^j$ there exists a constant $\mathbf{C} > 0$, there exists a neighborhood $V_{\mathbf{y}}$ of $\mathbf{y}$ and a neighborhood $V_{\vec{0}_d}$ of $\vec{0}_{\mathbb{R}^d}$ such that for all $\mathbf{x} \in D_0$ and for all $\lambda \in S^{j-1}$, for all $\mathbf{u} \in V_{\mathbf{y}}$ and for all $\mathbf{w} \in V_{\vec{0}_d}$ it holds $p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w}) \leqslant \mathbf{C}$.
Let us notice that this last conclusion holds true when the processes X and Y satisfy the condition $\mathbf{C_4}$.
In this case by using the Proposition 3.3.2, we shall deduce that for any compact set D_0 contained in D and for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D_0,\, X(\mathbf{x})(\omega) = \mathbf{y}\,, \text{rank}\, \nabla X(\mathbf{x})(\omega) < j\} = 0.$$

By choosing the compact $D_0 = D^{(n)} = \{\mathbf{x} \in \mathbb{R}^d,\, d(\mathbf{x}, D^c) \geqslant \frac{1}{n}\} \subseteq D$ we will deduce, since $D^{(n)}$ tends in a nondecreasing form towards D, when n tends to infinity, that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D,\, X(\mathbf{x})(\omega) = \mathbf{y}\,, \text{rank}\, \nabla X(\mathbf{x})(\omega) < j\} = 0.$$

We then shall have shown that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] \\ &\qquad = \mathbb{E}\left[\int_{\mathcal{C}_X^{D^r}(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] \\ &\qquad = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) H(\nabla X(\mathbf{x})) | X(\mathbf{x}) = \mathbf{y}\right] \mathrm{d}\mathrm{x}, \end{aligned}$$

that will end the proof of this theorem.
Thus let us verify that assumption $(\mathbf{S})$ holds.
Let D_0 be a compact set contained in D. For all $\mathbf{x} \in D_0$ and for all $\lambda \in S^{j-1}$, in the case where Y satisfies condition (3.18) and X and Y satisfy one of the conditions $\mathbf{C_i}$, $i = 1, 3$, as a first step we will study the density $p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})}$ of the vector $(X(\mathbf{x}),\, \lambda \cdot \nabla X(\mathbf{x}))$. We shall express this last one as function of the density $p_{X(\mathbf{x}), \nabla X(\mathbf{x})}$ of vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ that exists by the proof of Proposition 3.1.2 (c.f. equality (3.25)). So let us consider $\lambda \in S^{j-1}$, $\lambda = (\lambda_1, \cdots, \lambda_j)$. There exists a $k \in \{1, \cdots, j\}$ such that $|\lambda_k| \geqslant \frac{1}{\sqrt{j}}$. We will assume for instance $k = j$ and that $|\lambda_j| \geqslant \frac{1}{\sqrt{j}}$

this will imply $\frac{1}{|\lambda_j}| \leqslant \sqrt{j}$.
If $\mathbf{u} = (u_1, \cdots, u_j) \in \mathbb{R}^j$ and

$$\mathbf{s} = (s_{11}, s_{21}, \cdots, s_{j1}, s_{12}, s_{22}, \cdots, s_{j2}, \cdots, s_{1d}, s_{2d}, \cdots, s_{jd}) \in \mathbb{R}^{jd},$$

let us make as in the proof of the Proposition 3.1.2 (third part) the following change of variables. Let K be the function defined by

$$K : \mathbb{R}^j \times \mathbb{R}^{jd} \longrightarrow \mathbb{R}^j \times \mathbb{R}^d \times \mathbb{R}^{(j-1)d}$$
$$(\mathbf{u}; \mathbf{s}) \longmapsto \Big(\mathbf{u}; \sum_{i=1}^{j} \lambda_i s_{i1}, \sum_{i=1}^{j} \lambda_i s_{i2}, \cdots, \sum_{i=1}^{j} \lambda_i s_{id};$$
$$s_{11}, s_{21}, \cdots, s_{j-11}, s_{12}, s_{22}, \cdots, s_{j-12}, \cdots, s_{1d}, s_{2d}, \cdots, s_{j-1d}\Big)$$

The Jacobian J_K of this transformation is such that $\forall (\mathbf{t}, \mathbf{s}) \in \mathbb{R}^j \times \mathbb{R}^{jd}$

$$|J_K(\mathbf{u}, \mathbf{s})| = |\lambda_j|^d \neq 0,$$

by hypothesis.
Thus K is a bijection of class $C^1$and also its inverse K^{-1} given by

$$K^{-1} : \mathbb{R}^j \times \mathbb{R}^d \times \mathbb{R}^{(j-1)d} \longrightarrow \mathbb{R}^j \times \mathbb{R}^{jd}$$
$$\big(\mathbf{u}; s_{j1}, s_{j2}, .., s_{jd}; s_{11}, s_{21}, .., s_{j-11}, s_{12}, s_{22}, .., s_{j-12}, .., s_{1d}, s_{2d}, .., s_{j-1d}\big)$$
$$\longmapsto \Big(\mathbf{u}; s_{11}, s_{21}, \cdots, s_{j-11}, \frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{i1} + s_{j1}], s_{12}, s_{22}, \cdots, s_{j-12},$$
$$\frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{i2} + s_{j2}], \cdots, s_{1d}, s_{2d}, \cdots, s_{j-1d}, \frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{id} + s_{jd}]\Big)$$

For all $\lambda \in S^{j-1}$ for all $\mathbf{x} \in D_0$ we have

$$K(X(\mathbf{x}); \nabla X(\mathbf{x})) = (X(\mathbf{x}); \lambda \cdot \nabla X(\mathbf{x}); (\nabla X(\mathbf{x}))_{(j-1)d}),$$

where if $\mathbf{s} \in \mathbb{R}^{jd}$ we have denoted $\mathbf{s}_{(j-1)d}$ by

$$\mathbf{s}_{(j-1)d} = (s_{11}, s_{21}, \cdots, s_{j-11}, s_{12}, s_{22}, \cdots, s_{j-12}, \cdots, s_{1d}, s_{2d}, \cdots, s_{j-1d}).$$

With these notations if $p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x}), (\nabla X(\mathbf{x}))_{(j-1)d}}$ denotes the density of the vector $(X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x}), (\nabla X(\mathbf{x}))_{(j-1)d})$, we have: $\forall \lambda \in S^{j-1}$, $\forall \mathbf{x} \in D_0$, $\forall (\mathbf{u}, \mathbf{s}) \in \mathbb{R}^j \times \mathbb{R}^{jd}$,

$$p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x}), (\nabla X(\mathbf{x}))_{(j-1)d}}(\mathbf{u}, \mathbf{s}) = \frac{1}{|\lambda_j|^d} \; p_{X(\mathbf{x}), \nabla X(\mathbf{x})}(K^{-1}(\mathbf{u}, \mathbf{s})).$$

We deduce that

$$\forall \lambda \in S^{j-1}, \forall \mathbf{x} \in D_0, \forall (\mathbf{u}, \mathbf{w}) \in \mathbb{R}^j \times \mathbb{R}^d, \mathbf{w} = (w_1, w_2, \cdots, w_d),$$

$$p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w}) = \frac{1}{|\lambda_j|^d} \int_{\mathbb{R}^{(j-1)d}} p_{X(\mathbf{x}), \nabla X(\mathbf{x})}(\mathbf{u}; s_{11}, s_{21}, .., s_{j-11}, \frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{i1} + w_1], s_{12}, s_{22}, .., s_{j-12}, \frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{i2} + w_2], .., s_{1d}, s_{2d}, .., s_{j-1d}, \frac{1}{\lambda_j}[-\sum_{i=1}^{j-1} \lambda_i s_{id} + w_d]) d\mathbf{s}_{(j-1)d} \quad (3.36)$$

We now have to bound this density. In this aim let us consider each of $\mathbf{C_i}$, $i = 1, 3$ conditions.

- If X and Y satisfy the condition $\mathbf{C_1}$, then for all $\mathbf{x} \in D_0$ the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ has a non singular density and since X is a process of class C^2 the covariance matrix of this vector is strictly positive on the compact set D_0 . Then there exist reals $a, b > 0$ such that for all $\mathbf{x} \in D_0$, $0 < a \leqslant \inf_{||z||_{j(d+1)}=1} ||\mathbb{V}(X(\mathbf{x}), \nabla X(\mathbf{x})) \times z||_{j(d+1)} \leqslant b$. In the same form that we have obtained the equality (3.27), we get with the same notations as before that there exists a number $\mu > 0$ and a number $\mathbf{C} > 0$, such that for all $(\mathbf{x}, \mathbf{u}, \mathbf{s}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj}$,

$$\begin{aligned} \mathrm{p}_{X(\mathbf{x}), \nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{s}) &\leqslant \mathbf{C}\, e^{-\mu ||(\mathbf{u}, \mathbf{s})||^2_{j(d+1)}} \leqslant \mathbf{C}\, e^{-\mu ||\mathbf{s}||^2_{jd}} \\ &\leqslant \mathbf{C}\, e^{-\mu ||\mathbf{s_{(j-1)d}}||^2_{(j-1)d}}. \end{aligned}$$

  By using the equality (3.36), we obtain the following bound: $\forall \lambda \in S^{j-1}, \forall \mathbf{x} \in D_0, \forall (\mathbf{u}, \mathbf{w}) \in \mathbb{R}^j \times \mathbb{R}^d$,

$$\begin{aligned} p_{X(\mathbf{x}), \lambda \cdot \nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{w}) &\leqslant \frac{\mathbf{C}}{|\lambda_j|^d} \int_{\mathbb{R}^{(j-1)d}} e^{-\mu ||\mathbf{s_{(j-1)d}}||^2_{(j-1)d}} \, d\mathbf{s_{(j-1)d}} \\ &\leqslant \mathbf{C}, \end{aligned}$$

  the last inequality comes from the fact that $\frac{1}{|\lambda_j|^d} \leqslant (\sqrt{j})^d$.

- If X and Y satisfy the condition $\mathbf{C_2}$, then for all $\mathbf{x} \in D_0$ the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ has a non-degenerate density. In the same form that in the first part we show the existence of a number $\mu > 0$ and of a number $\mathbf{C} > 0$ such that for all $(\mathbf{x}, \mathbf{u}, \mathbf{s}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj}$,

$$\mathrm{p}_{Z(\mathbf{x}),\nabla Z(\mathbf{x})}(\mathbf{u}, \mathbf{s}) \leqslant \mathbf{C}\, e^{-\mu||\mathbf{s}||_{jd}^2}.$$

Moreover, the equality (3.25) proved in the third part of the proof of the Proposition 3.1.2 and applied to $j = j'$ shows that the density of the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ is given by: for all $(\mathbf{x}, \mathbf{u}, \mathbf{s}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj}$,

$$\begin{aligned}\mathrm{p}_{X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{s}) &= \frac{1}{|J_F(F^{-1}(\mathbf{u}))|^{d+1}} \times \\ &\quad \mathrm{p}_{Z(\mathbf{x}),\nabla Z(\mathbf{x})}(F^{-1}(\mathbf{u}), (\nabla F(F^{-1}(\mathbf{u})))^{-1} \times \mathbf{s}).\end{aligned}$$

We deduce that there exists a constant $\mu > 0$ such that for all $(\mathbf{x}, \mathbf{u}, \mathbf{s}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj}$,

$$\mathrm{p}_{X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{u}, \mathbf{s}) \leqslant \frac{\mathbf{C}}{|J_F(F^{-1}(\mathbf{u}))|^{d+1}} \times e^{-\mu||(\nabla F(F^{-1}(\mathbf{u})))^{-1} \times \mathbf{s}||_{jd}^2}.$$

Now let $\mathbf{y}$ be a vector fixed in $\mathbb{R}^j$. Since the function F belongs to C^1 and F^{-1} is continuous the Jacobian $J_F(F^{-1})$ is continuous on $\mathbb{R}^j$ and it is everywhere non zero. Let $V_{\mathbf{y}}$ a compact neighborhood of $\mathbf{y}$, then there exists a $\mathbf{C} > 0$ such that for all $\mathbf{u} \in V_{\mathbf{y}}$ we have $\frac{\mathbf{1}}{|J_F(F^{-1}(\mathbf{u}))|^{d+1}} \leqslant \mathbf{C}$.
Moreover, for all $\mathbf{u} \in \mathbb{R}^j$, $(\nabla F(F^{-1}(\mathbf{u})))^{-1} \in \mathfrak{L}(\mathbb{R}^j, \mathbb{R}^j)$ and thus for all $\mathbf{s} \in \mathbb{R}^{jd}$,

$$||(\nabla F(F^{-1}(\mathbf{u})))^{-1} \times \mathbf{s}||_{jd} \geqslant \frac{||\mathbf{s}||_{jd}}{||\nabla F(F^{-1}(\mathbf{u}))||_{j,j}}. \tag{3.37}$$

Since F belongs to C^1 and F^{-1} is continuous, the operator $\nabla F(F^{-1}(\cdot))$ is a continuous function of $\mathbb{R}^j$ into $\mathfrak{L}(\mathbb{R}^j, \mathbb{R}^j)$. There exists a constant $\mathbf{C} > 0$, such that for all $\mathbf{u} \in V_{\mathbf{y}}$, we have

$$||\nabla F(F^{-1}(\mathbf{u}))||_{j,j} \leqslant \mathbf{C}.$$

We deduce that for all $\mathbf{s} \in \mathbb{R}^{jd}$ and for all $\mathbf{u} \in V_{\mathbf{y}}$,

$$||(\nabla F(F^{-1}(\mathbf{u})))^{-1} \times \mathbf{s}_{j,d}||_{jd} \geqslant \mathbf{C}\,||\mathbf{s}||_{jd}.$$

Finally we conclude that for all $\mathbf{y} \in \mathbb{R}^j$, there exists a constant $\mathbf{C} > 0$, there exists a neighborhood $V_{\mathbf{y}}$ of $\mathbf{y}$, there exists a constant $\mu > 0$ such that for all $\mathbf{x} \in D_0$, for all $\mathbf{u} \in V_{\mathbf{y}}$ and for all $\mathbf{s} \in \mathbb{R}^{jd}$ we have

$$\mathrm{p}_{X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{u},\mathbf{s}) \leqslant \mathbf{C}\, e^{-\mu||\mathbf{s}||^2_{jd}}.$$

Then in the same form that in the first part of the proof of this theorem we deduce that for all $\mathbf{y} \in \mathbb{R}^j$, there exists a constant $\mathbf{C} > 0$, there exists a neighborhood of $\mathbf{y}$, let $V_{\mathbf{y}}$, such that for all $\mathbf{x} \in D_0$ and for all $\lambda \in S^{j-1}$, for all $\mathbf{u} \in V_{\mathbf{y}}$ and for all $\mathbf{w} \in \mathbb{R}^d$,

$$p_{X(\mathbf{x}),\,\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u},\mathbf{w}) \leqslant \mathbf{C}.$$

- If X and Y satisfy the condition $\mathbf{C_3}$, in the same form as before and since D_0 is a compact set, there exist constants $\mu > 0$ and $\mathbf{C} > 0$ such that for all $(\mathbf{x},\mathbf{u},\mathbf{s}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj'}$,

$$\mathrm{p}_{Z(\mathbf{x}),\nabla Z(\mathbf{x})}(\mathbf{u},\mathbf{s}) \leqslant \mathbf{C}\; e^{-\mu||\mathbf{u}||^2_{j'}}\; e^{-\mu||\mathbf{s}||^2_{j'd}}.$$

The equality (3.25) proved in the third part of the proof of Proposition 3.1.2, shows that the density of the random vector $(X(\mathbf{x}),\nabla X(\mathbf{x}))$ exists. Using the same notations that in this proof and by the equality given in (3.37), we can prove that this last density is bounded in the following form, for all

$$(\mathbf{x},\mathbf{u},\mathbf{s}_{j,d}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj},$$

$$\mathrm{p}_{X(\mathbf{x});\nabla X(\mathbf{x})}(\mathbf{u};\mathbf{s_{j,d}}) \leqslant \mathbf{C}\int_{\mathbb{R}^{j'-j}}\int_{\mathbb{R}^{(j'-j)d}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} \times$$
$$e^{-\mu||\mathbf{z}||^2_{j'-j}} \times e^{-\mu||\mathbf{s_{j'-j,d}}||^2_{(j'-j)d}} \times$$
$$e^{-\mu\,||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||^{-2}_{jj}\times||\mathbf{s_{j,d}}-\left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{jj'-j}\mathbf{s_{j'-j,d}}||^2_{jd}}\, d\mathbf{s_{j'-j,d}}\, d\mathbf{z}.$$

We deduce that for all $(\mathbf{x},\mathbf{u},\mathbf{s}_{j,d}) \in D_0 \times \mathbb{R}^j \times \mathbb{R}^{dj}$,

$$p_{X(\mathbf{x});\nabla X(\mathbf{x})}(\mathbf{u};\mathbf{s_{j,d}}) \leqslant \mathbf{C}\int_{\mathbb{R}^{j'-j}}\int_{\mathbb{R}^{(j'-j)d}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} \times$$
$$e^{-\mu||\mathbf{z}||^2_{j'-j}} \times e^{-\mu||\mathbf{s_{j'-j,d}}||^2_{(j'-j)d}} \times$$
$$e^{-\mu\,||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||^{-2}_{jj}\times||\mathbf{s_{j-1,d}}-\left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{j-1j'-j}\mathbf{s_{j'-j,d}}||^2_{j-1d}} d\mathbf{s_{j'-j,d}}d\mathbf{z},$$

where the matrix $\mathbf{s_{j-1,d}}$ is the matrix $\mathbf{s_{j,d}}$ of which we have deleted the j-th row and $\left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{j-1j'-j}$ is the matrix $\left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{jj'-j}$ of which we have deleted the j-th row.
By using the equality (3.36), we get:
$\exists\,\mathbf{C}>0,\ \exists\,\mu>0,\ \forall\mathbf{x}\in D_0,\ \forall\lambda\in S^{j-1},\ \forall(\mathbf{u},\mathbf{w})\in\mathbb{R}^j\times\mathbb{R}^d$,

$$p_{X(\mathbf{x}),\,\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u},\mathbf{w}) \leqslant \mathbf{C}\int_{\mathbb{R}^{(j-1)d}\times\mathbb{R}^{j'-j}\times\mathbb{R}^{(j'-j)d}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}}$$
$$\times e^{-\mu||\mathbf{z}||^2_{j'-j}} \times e^{-\mu||\mathbf{s_{j'-j,d}}||^2_{(j'-j)d}} \times$$
$$e^{-\mu\,||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||^{-2}_{jj}||\mathbf{s_{j-1,d}}-\left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{j-1j'-j}\mathbf{s_{j'-j,d}}||^2_{j-1d}}$$
$$d\mathbf{s_{j'-j,d}}d\mathbf{z}d\mathbf{s_{j-1,d}}.$$

We make the following change of variables in the integral over $\mathbb{R}^{(j-1)d}$:

$$\mathbf{s_{j-1,d}} - \left[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})\right]_{j-1j'-j}\mathbf{s_{j'-j,d}}$$
$$= ||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||_{jj}\cdot\mathbf{v_{j-1,d}}.$$

We get: $\exists\,\mathbf{C}>0,\ \exists\,\mu>0,\ \forall\mathbf{x}\in D_0,\ \forall\lambda\in S^{j-1},\ \forall(\mathbf{u},\mathbf{w})\in\mathbb{R}^j\times\mathbb{R}^d$,

$$p_{X(\mathbf{x}),\,\lambda\cdot\nabla X(\mathbf{x})}(\mathbf{u},\mathbf{w})$$
$$\leqslant \mathbf{C}\int_{\mathbb{R}^{(j-1)d}}\int_{\mathbb{R}^{j'-j}}\int_{\mathbb{R}^{(j'-j)d}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} \times$$
$$e^{-\mu||\mathbf{z}||^2_{j'-j}} \times e^{-\mu||\mathbf{s_{j'-j,d}}||^2_{(j'-j)d}} \times$$
$$e^{-\mu\,||\mathbf{v_{j-1,d}}||^2_{(j-1)d}} \times ||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||^{(j-1)d}_{jj}\,d\mathbf{s_{j'-j,d}}d\mathbf{z}d\mathbf{v_{j-1,d}} \leqslant$$
$$\mathbf{C}\int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} e^{-\mu||\mathbf{z}||^2_{j'-j}} ||[\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})]_{jj}||^{(j-1)d}_{jj}\,d\mathbf{z}$$
$$\leqslant \mathbf{C}\int_{\mathbb{R}^{j'-j}} \frac{1}{|J_F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})|^{d+1}} e^{-\mu||\mathbf{z}||^2_{j'-j}} ||\nabla F(F_{\mathbf{z}}^{-1}(\mathbf{u}),\mathbf{z})||^{(j-1)d}_{jj'}\,d\mathbf{z}.$$

The $\mathbf{C_3}$ condition allows getting that for all $\mathbf{y} \in \mathbb{R}^j$, there exists $\mathbf{C} > 0$, a neighborhood $V_{\mathbf{y}}$ of $\mathbf{y}$, such that $\forall \mathbf{x} \in D_0$, $\forall \lambda \in S^{j-1}$, $\forall \mathbf{u} \in V_{\mathbf{y}}$ and $\forall \mathbf{w} \in \mathbb{R}^d$, we have

$$p_{X(\mathbf{x}),\lambda \cdot \nabla X(\mathbf{x})}(\mathbf{u},\mathbf{w}) \leqslant \mathbf{C}.$$

This ends the proof of the theorem.

□

In the same manner as in Theorem 3.2.2, we can free ourselves in the Theorem 3.3.1, from the assumption $\mathbb{E}(L_X(\cdot))^d < +\infty$. If one of the conditions $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$ or $\mathbf{C_4}$ is replaced by $\mathbf{C_1^\star}$, $\mathbf{C_2^\star}$, $\mathbf{C_3^\star}$ or $\mathbf{C_4^\star}$, the Rice's formula will be still true. In the three first conditions $\mathbf{C_1^\star}$, $\mathbf{C_2^\star}$ and $\mathbf{C_3^\star}$, we make the hypothesis that Y can be written under the form (3.18).
More precisely

- $\mathbf{C_1^\star}$: It is the condition $\mathbf{C_1}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$ the density of the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}))$ exists.

- $\mathbf{C_2^\star}$: It is the condition $\mathbf{C_2}$ plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$ the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}))$ exists.

- $\mathbf{C_3^\star}$: It is the condition $\mathbf{C_3}$ plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$ the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}))$ exists.

- $\mathbf{C_4^\star}$: It is the condition $\mathbf{B_4^\star}$ plus the following hypothesis. The process X verifies assumption $(\mathbf{S})$.

The following theorem synthesizes all the results which we obtained previously. In a certain sense one can say that it is a new result. Let us state it.

Theorem 3.3.2 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) be a random field belonging to $\mathcal{C}^2(D, \mathbb{R}^j)$, where D is a convex open bounded set of $\mathbb{R}^d$, such that for almost surely $\omega \in \Omega$, $\nabla X(\omega)$ is Lipschitz. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous process. If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{C_i^\star}$, $i = 1, 2, 3$ and the hypotheses*

$\mathbf{H_6}$ *and* $\mathbf{H_6^\star}$ *or if X and Y satisfy the condition* $\mathbf{C_4^\star}$*, then for all* $\boldsymbol{y} \in \mathbb{R}^j$ *we have*

$$\mathbb{E}\left[\int_{\mathcal{C}_X(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]$$
$$= \int_D p_{X(\boldsymbol{x})}(\boldsymbol{y})\, \mathbb{E}\left[Y(\boldsymbol{x}) H(\nabla X(\boldsymbol{x})) | X(\boldsymbol{x}) = \boldsymbol{y}\right] dx.$$

Remark 3.3.3 In the same form as in the Remark 3.3.1, we can replace in the theorem the hypothesis "for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz", by the hypothesis " almost surely
$L_X = \sup_{\mathbf{x} \in D} ||\nabla^2 X(\mathbf{x})||_{j,d}^{(s)} < +\infty$ ", since almost surely the process ∇X will be Lipschitz with Lipschitz constant L_X.

Remark 3.3.4 We can generalized this theorem in the case where D is a convex open not necessarily bounded. We maintain the same hypotheses for the process X adapting those of Y that is now defined on the bounded open set D_1 included in D instead of D and to X/D_1.
The Rice's formula will be still valid for all level $\mathbf{y} \in \mathbb{R}^j$ and for X/D_1 and Y defined on D_1.

3.3.3 Rice formula for the k-th moment

The Theorem 3.3.1 will allow to state a general Rice formula for the second moment.
Let us set the conditions $\mathbf{D}_i$, $i = 1, 4$, where in the three first ones $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$, we will assume the hypothesis that Y can be written in the form (3.18).
We will denote Δ the subset of $\mathbb{R}^{2d}$, $\Delta = \{(\mathbf{x_1}, \mathbf{x_2}) \in D \times D,\ \mathbf{x_1} = \mathbf{x_2}\}$, where D is an open set of $\mathbb{R}^d$. Let us state the following conditions $\mathbf{D_i}$, $i = 1, 4$.

- $\mathbf{D_1}$: It is the condition $\mathbf{E_1}$, plus the following hypothesis. For all $\mathbf{x} \in D$, the vector $(X(\mathbf{x}), \nabla X(\mathbf{x}))$ has a density.

- $\mathbf{D_2}$: It is the condition $\mathbf{E_2}$, plus the following hypothesis. For all $\mathbf{x} \in D$, the vector $(Z(\mathbf{x}), \nabla Z(\mathbf{x}))$ has a density.

- $\mathbf{D_3}$: It is the condition $\mathbf{E_3}$, plus the following hypothesis. The function F verifies assumption $(\mathbf{FF})$ appearing in condition $\mathbf{C_3}$.

- $\mathbf{D_4}$: It is the condition $\mathbf{E_4}$, plus the following hypothesis. The process X satisfies the assumption $(\mathbf{S})$.

The conditions $\mathbf{E_i}$, $i = 1, 4$, are the following:

- $\mathbf{E_1}$: The process $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is Gaussian of class C^2 on D, such that for all $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D - \Delta$, the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}))$ has a density. Moreover, for almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the vector $(W(\mathbf{x_1}), W(\mathbf{x_2}))$ is independent of the vector
$(X(\mathbf{x_1}), X(\mathbf{x_2}), \nabla X(\mathbf{x_1}), \nabla X(\mathbf{x_2}))$, and $\forall\, n \in \mathbb{N}$,
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\, d\mathbf{x} < +\infty.$$

- $\mathbf{E_2}$: $\forall\, \mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where $F : \mathbb{R}^j \longrightarrow \mathbb{R}^j$ is a bijective function of class C^2, such that $\forall\, \mathbf{z} \in \mathbb{R}^j$, the Jacobian of F in $\mathbf{z}$, that is $J_F(\mathbf{z})$ is such that $J_F(\mathbf{z}) \neq 0$ and the function F^{-1} is continuous. The process $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ ($j \leqslant d$) is Gaussian of class C^2 on D, such that for all $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D - \Delta$, the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}))$ has a density. Moreover, for almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the vector $(W(\mathbf{x_1}), W(\mathbf{x_2}))$ is independent of the vector
$(Z(\mathbf{x_1}), Z(\mathbf{x_2}), \nabla Z(\mathbf{x_1}), \nabla Z(\mathbf{x_2}))$, and $\forall\, n \in \mathbb{N}$,
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\; d\mathbf{x} < +\infty.$$

- $\mathbf{E_3}$: $\forall\, \mathbf{x} \in D$, $X(\mathbf{x}) = F(Z(\mathbf{x}))$, where the process $Z : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^{j'}$ is Gaussian of class C^2 on D, such that for all $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D - \Delta$, the vector
$$(Z(\mathbf{x_1}), Z(\mathbf{x_2}), \nabla Z(\mathbf{x_1}), \nabla Z(\mathbf{x_2}))$$
has a density. Moreover, for almost surely $(\mathbf{x_1}, \mathbf{x_2}) \in D \times D$, the vector $(W(\mathbf{x_1}), W(\mathbf{x_2}))$ is independent of the vector
$(Z(\mathbf{x_1}), Z(\mathbf{x_2}), \nabla Z(\mathbf{x_1}), \nabla Z(\mathbf{x_2}))$.
And, $\forall\, n \in \mathbb{N}$,
$$\int_D \mathbb{E}(||W(\mathbf{x})||_k^n)\; d\mathbf{x} < +\infty.$$
The function F verifies assumption $(\mathbf{F})$ appearing in condition $\mathbf{A_3}$.

- $\mathbf{E_4}$: For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{y_1}, \mathbf{y_2}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2) \in$ $D \times D \times \mathbb{R}^2 \times \mathbb{R}^{dj} \times \mathbb{R}^{dj}$ and for all $\mathbf{u} \in \mathbb{R}^j$, the density

$$\mathrm{p}_{Y(\mathbf{x_1}),Y(\mathbf{x_2}),X(\mathbf{x_1}),X(\mathbf{x_2}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2})}(\mathbf{y_1}, \mathbf{y_2}, \mathbf{u}, \mathbf{u}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_1),$$

of the joint distribution of

$$(Y(\mathbf{x_1}), Y(\mathbf{x_2}), X(\mathbf{x_1}), X(\mathbf{x_2}), \nabla X(\mathbf{x_1}), \nabla X(\mathbf{x_2})),$$

exists and is continuous in the variable $\mathbf{u}$.
Furthermore

$$\mathbf{u} \longmapsto \int_{D\times D} \int_{\mathbb{R}^2\times\mathbb{R}^{2dj}} |\mathbf{y_1}|\,|\mathbf{y_2}|\,||\dot{\mathbf{x}}_1||_{dj}^{j}\,||\dot{\mathbf{x}}_2||_{dj}^{j} \times$$

$$\mathrm{p}_{Y(\mathbf{x_1}),Y(\mathbf{x_2}),X(\mathbf{x_1}),X(\mathbf{x_2}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2})}(\mathbf{y_1}, \mathbf{y_2}, \mathbf{u}, \mathbf{u}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_1) \, d\dot{\mathbf{x}}_1 d\dot{\mathbf{x}}_2 d\mathbf{y_1} d\mathbf{y_2} d\mathbf{x_1} d\mathbf{x_2},$$

is continuous.

Let us state the hypothesis $\mathbf{H_7}$.

- $\mathbf{H_7}$: For all $\mathbf{y} \in \mathbb{R}^j$,

$$\int_{D\times D} \mathbb{E}[|Y(\mathbf{x}_1)||Y(\mathbf{x}_2)|H(\nabla X(\mathbf{x}_1))H(\nabla X(\mathbf{x}_2))|X(\mathbf{x}_1) = X(\mathbf{x}_2) = \mathbf{y}]$$

$$\times \mathrm{p}_{X(\mathbf{x}_1),X(\mathbf{x}_2)}(\mathbf{y}, \mathbf{y})\, d\mathbf{x_1}\, d\mathbf{x_2} < +\infty.$$

We are ready to prove the following theorem.

Theorem 3.3.3 *Let $X : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ $(j < d)$ be a random field belonging to $\boldsymbol{C}^2(D, \mathbb{R}^j)$, where D is a bounded, convex open set of $\mathbb{R}^d$, such that for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^{2d} < +\infty$. Let $Y : \Omega \times D \subset \Omega \times \mathbb{R}^d \to \mathbb{R}$ a continuous process.*
If Y satisfies the condition (3.18) and if X and Y satisfy one of the three conditions $\mathbf{D_i}$, $i = 1, 2, 3$ and the hypothesis $\mathbf{H_7}$ or if X and Y satisfy the condition $\mathbf{D_4}$, then for all $\boldsymbol{y} \in \mathbb{R}^j$ we have

$$\mathbb{E}\left[\int_{\mathcal{C}_X(\boldsymbol{y})} Y(\boldsymbol{x}) d\sigma_{d-j}(\boldsymbol{x})\right]^2 =$$

$$\int_{D\times D} \mathbb{E}[Y(\boldsymbol{x}_1)Y(\boldsymbol{x}_2)H(\nabla X(\boldsymbol{x}_1))H(\nabla X(\boldsymbol{x}_2))|X(\boldsymbol{x}_1) = X(\boldsymbol{x}_2) = \boldsymbol{y}]$$

$$\times p_{X(\boldsymbol{x}_1),X(\boldsymbol{x}_2)}(\boldsymbol{y}, \boldsymbol{y})\, d\mathbf{x_1}\, d\mathbf{x_2}. \quad (3.38)$$

Remark 3.3.5 In the same form as in the Remark 3.3.1, we can replace in the theorem the hypothesis " for almost surely $\omega \in \Omega$, $\nabla X(\omega)$ is Lipschitz with Lipschitz constant $L_X(\omega)$ such that $\mathbb{E}(L_X(\cdot))^{2d} < +\infty$ ", by the hypothesis " $\mathbb{E}(\sup_{\mathbf{x}\in D} ||\nabla^2 X(\mathbf{x})||_{j,d}^{(s)})^{2d} < +\infty$".

Remark 3.3.6 Under the same hypotheses as that of the theorem or under below those of the Remark 3.3.9, for $j = d$, we get a result similar to the one obtained in the equality (3.38). It is enough to replace

$$\mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right]^2,$$

by

$$\mathbb{E}\left[\left(\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right)^2 - \int_{\mathcal{C}_X(\mathbf{y})} Y^2(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right]$$

in the equality (3.38). The right hand side remains unchanged. However, we need to point out that is this particular case σ_0 is the counting measure.

Remark 3.3.7 Under the same type of hypotheses as those given in the theorem or farther in the Remark 3.3.9, we can propose a general Rice formula for the k order moments of the process Y integrated on the level set of the random field X, and this for all $\mathbf{y} \in \mathbb{R}^j$.

Remark 3.3.8 The Theorem 3.3.3 and also the Remarks 3.3.6, 3.3.7 and 3.3.9 can be generalized to D a convex open set $\mathbb{R}^d$ not necessarily bounded.
For this mutatis mutandi we can argue as in Remark 3.3.4.

Proof of Theorem 3.3.3. The idea consists in applying the Remark 3.2.3 following the Theorem 3.2.1 for the convex and open set $D \times D$ and the open bounded set $D_1 = D \times D - \Delta$, to the processes $\widetilde{X}$ and $\widetilde{Y}$ defined of the following form

$$\begin{aligned}\widetilde{X} : \Omega \times D \times D \subset \Omega \times \mathbb{R}^{2d} &\longrightarrow \mathbb{R}^{2j} \\ \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) &\longmapsto \widetilde{X}(\mathbf{x}) = (X(\mathbf{x_1}), X(\mathbf{x_2})),\end{aligned}$$

and

$$\widetilde{Y} : \Omega \times D \times D \subset \Omega \times \mathbb{R}^{2d} \longrightarrow \mathbb{R}$$
$$\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \longmapsto \widetilde{Y}(\mathbf{x}) = Y(\mathbf{x_1}) \times Y(\mathbf{x_2}).$$

Since X is a random field belonging to $C^2(D, \mathbb{R}^j)$ then $\widetilde{X}$ is a random field belonging to $C^2(D \times D, \mathbb{R}^{2j})$ and $D \times D$ is a convex and open set of $\mathbb{R}^{2d}$. Also $\widetilde{Y}/D \times D - \Delta$ is still continuous on $D \times D - \Delta$ open and bounded set of $\mathbb{R}^{2d}$ contained in $D \times D$.
Since for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla X(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_X(\boldsymbol{\omega})$ such that $\mathbb{E}(L_X(\cdot))^{2d} < +\infty$, then for almost surely $\boldsymbol{\omega} \in \Omega$, $\nabla\widetilde{X}(\boldsymbol{\omega})$ is Lipschitz with Lipschitz constant $L_{\widetilde{X}}(\boldsymbol{\omega}) = L_X(\boldsymbol{\omega})$, such that $\mathbb{E}(L_{\widetilde{X}}(\cdot))^{2d} < +\infty$.
Then under one of the conditions $\mathbf{C_i}$, $i = 1, 3$, since Y is written as a function G of X and of ∇X and of the variable $W : \Omega \times D \subset \mathbb{R}^d \to \mathbb{R}^k$, $k \in \mathbb{N}^\star$, in the following form, for almost surely $\mathbf{x} \in D$:

$$Y(\mathbf{x}) = G(\mathbf{x}, W(\mathbf{x}), X(\mathbf{x}), \nabla X(\mathbf{x})),$$

where

$$G : D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j) \longrightarrow \mathbb{R}$$
$$(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \longmapsto G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}),$$

is a continuous function of its variables over
$D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$ and such that $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A}) \in D \times \mathbb{R}^k \times \mathbb{R}^j \times \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$,

$$|G(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{A})| \leqslant P(f(\mathbf{x}), ||\mathbf{z}||_k, h(\mathbf{u}), ||\mathbf{A}||_{j,d}),$$

where P is a polynomial with positive coefficients and $f : D \longrightarrow \mathbb{R}^+$ and $h : \mathbb{R}^j \longrightarrow \mathbb{R}^+$ are continuous functions, the same holds true for $\widetilde{Y}$. More precisely, for almost surely $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in D \times D$,

$$\widetilde{Y}(\mathbf{x}) = \widetilde{G}(\mathbf{x}, \widetilde{W}(\mathbf{x}), \widetilde{X}(\mathbf{x}), \nabla\widetilde{X}(\mathbf{x})),$$

where

$$\widetilde{W} : \Omega \times D \times D \subset \Omega \times \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2k}$$
$$\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \longmapsto (W(\mathbf{x_1}), W(\mathbf{x_2})),$$

$$\begin{aligned}
\widetilde{G} : D^2 \times \mathbb{R}^{2k} \times \mathbb{R}^{2j} \times \mathfrak{B}(\mathbb{R}^{2d}, \mathbb{R}^{2j}) &\longrightarrow \mathbb{R} \\
(\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}), \mathbf{z} = (\mathbf{z_1}, \mathbf{z_2}), \mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}), \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}) & \\
&\longmapsto \widetilde{G}(\mathbf{x}, \mathbf{z}, \mathbf{u}, \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}) \\
&= G(\mathbf{x_1}, \mathbf{z_1}, \mathbf{u_1}, \mathbf{A}) \times G(\mathbf{x_2}, \mathbf{z_2}, \mathbf{u_2}, \mathbf{B}),
\end{aligned}$$

where $\mathfrak{B}(\mathbb{R}^{2d}, \mathbb{R}^{2j})$ is the vector subspace of $\mathfrak{L}(\mathbb{R}^{2d}, \mathbb{R}^{2j})$ of the matrices of the form $\mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$ where $\mathbf{A}, \mathbf{B} \in \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^j)$.
It is clear that $\widetilde{G}$ remains a continuous function defined on

$$D^2 \times \mathbb{R}^{2k} \times \mathbb{R}^{2j} \times \mathfrak{B}(\mathbb{R}^{2d}, \mathbb{R}^{2j}),$$

and it is such that $\forall\, (\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{C}) \in D^2 \times \mathbb{R}^{2k} \times \mathbb{R}^{2j} \times \mathfrak{B}(\mathbb{R}^d, \mathbb{R}^j)$,

$$\begin{aligned}
|\widetilde{G}(\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{C})| &\leqslant P(f(\mathbf{x_1}), ||\mathbf{z_1}||_k, h(\mathbf{u_1}), ||\mathbf{A}||_{j,d}) \\
&\qquad \times P(f(\mathbf{x_2}), ||\mathbf{z_2}||_k, h(\mathbf{u_2}), ||\mathbf{B}||_{j,d}) \\
&\leqslant \widetilde{P}(\widetilde{f}(\mathbf{x}), ||\mathbf{z}||_{2k}, \widetilde{h}(\mathbf{u}), ||\mathbf{C}||_{2j,2d}),
\end{aligned}$$

where the function $\widetilde{f}$ is defined by

$$\begin{aligned}
\widetilde{f} : D^2 &\longrightarrow \mathbb{R}^+ \\
\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) &\longmapsto \widetilde{f}(\mathbf{x}) = f(\mathbf{x_1}) + f(\mathbf{x_2}),
\end{aligned}$$

and the function $\widetilde{h}$ is defined by

$$\begin{aligned}
\widetilde{h} : \mathbb{R}^{2j} &\longrightarrow \mathbb{R}^+ \\
\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}) &\longmapsto \widetilde{h}(\mathbf{u}) = h(\mathbf{u_1}) + h(\mathbf{u_2}),
\end{aligned}$$

that are still continuous functions and $\widetilde{P}$ is a polynomial with positive coefficients.

It is easy to check that $\widetilde{X}$ and $\widetilde{Y}/D \times D - \Delta$ satisfy the hypotheses $\mathbf{B_i}$, $i = 1, 4$, of the Remark 3.2.3 following the Theorem 3.2.1, respectively for the convex open set $D \times D$ and for the open and bounded set $D \times D - \Delta$ contained in $D \times D$.

Furthermore, if X and Y satisfy the hypothesis $\mathbf{H_7}$ then $\widetilde{X}$ et $\widetilde{Y}/D \times D - \Delta$ satisfy the hypothesis $\mathbf{H_6}$, since $\forall \mathbf{x} \in D \times D$,

$$H(\nabla \widetilde{X}(\mathbf{x})) = H(\nabla X(\mathbf{x_1})) \times H(\nabla X(\mathbf{x_2})). \tag{3.39}$$

It holds that $\widetilde{X}$ and $\widetilde{Y}/D \times D - \Delta$ satisfy the hypotheses of the Remark 3.2.3 following the Theorem 3.2.1.
Now the hypotheses that X and Y satisfy make that these two processes verify hypotheses $\mathbf{C_i}$, $i = 1, 4$, appearing in Theorem 3.3.1 and then the ones of the Proposition 3.3.2. Hence in the same form as in this theorem we get that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D,\ X(\mathbf{x})(\omega) = \mathbf{y}\,, \text{rank}\, \nabla X(\mathbf{x})(\omega) < j\} = 0. \tag{3.40}$$

Deducing for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{P}\{\omega \in \Omega : \exists \mathbf{x} \in D \times D, \widetilde{X}(\mathbf{x})(\omega) = (\mathbf{y}, \mathbf{y}), \text{rank} \nabla \widetilde{X}(\mathbf{x})(\omega) < 2j\} = 0. \tag{3.41}$$

Indeed by using the equality (3.39), for all $\mathbf{y} \in \mathbb{R}^j$,

$$\begin{aligned} &\mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D \times D,\ \widetilde{X}(\mathbf{x})(\omega) = (\mathbf{y}, \mathbf{y})\,, \text{rank}\, \nabla \widetilde{X}(\mathbf{x})(\omega) < 2j\} \\ &\quad \leqslant \mathbb{P}\{\omega \in \Omega : \exists\, \mathbf{x} \in D,\ X(\mathbf{x})(\omega) = \mathbf{y}\,, \text{rank}\, \nabla X(\mathbf{x})(\omega) < j\} = 0. \end{aligned}$$

The Remark 3.2.3 following the Theorem 3.2.1 applied to $\widetilde{X}$ and $\widetilde{Y}/D \times D - \Delta$ and the equality (3.41) allow writing for all $\mathbf{y} \in \mathbb{R}^j$,

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathcal{C}_{D\times D-\Delta,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] \\ &= \mathbb{E}\left[\int_{\mathcal{C}_{D\times D-\Delta,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] \\ &= \int_{D\times D-\Delta} \mathrm{p}_{\widetilde{X}(\mathbf{x})}(\mathbf{y},\mathbf{y})\, \mathbb{E}\left[\widetilde{Y}(\mathbf{x}) H(\nabla \widetilde{X}(\mathbf{x})) | \widetilde{X}(\mathbf{x}) = (\mathbf{y},\mathbf{y})\right] \mathrm{dx} \\ &= \int_{D\times D} \mathbb{E}[Y(\mathbf{x}_1) Y(\mathbf{x}_2) H(\nabla X(\mathbf{x}_1)) H(\nabla X(\mathbf{x}_2)) | X(\mathbf{x}_1) = X(\mathbf{x}_2) = \mathbf{y}] \\ &\qquad\qquad\qquad \times \mathrm{p}_{X(\mathbf{x}_1),X(\mathbf{x}_2)}(\mathbf{y},\mathbf{y})\, d\mathbf{x_1}\, d\mathbf{x_2}, \end{aligned}$$

the last equality comes from the fact that $\sigma_{2d}(\Delta) = 0$.
Moreover, we know by using the Remark 3.1.2 and the equality (3.40) that for all $\mathbf{y} \in \mathbb{R}^j$, almost surely $\mathcal{C}_X^{D^r}(\mathbf{y}) = \mathcal{C}_X(\mathbf{y})$ and $\mathcal{C}_X^{D^r}(\mathbf{y})$ is a differentiable manifold of dimension $(d - j)$. Thus for all $\mathbf{y} \in \mathbb{R}^j$, almost

surely the set A defined by $A = \{(\mathbf{x}, \mathbf{x}) \in D \times D,\ X(\mathbf{x}) = \mathbf{y}\}$ is a differentiable manifold of dimension $(d-j)$ then, since $j < d$, almost surely $\sigma_{2(d-j)}(A) = 0$. Thus for all $\mathbf{y} \in \mathbb{R}^j$,

$$\begin{aligned}\mathbb{E}\left[\int_{\mathcal{C}_{D\times D-\Delta,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] &= \mathbb{E}\left[\int_{\mathcal{C}_{D\times D,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] \\ &= \mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right]^2,\end{aligned}$$

the last equality comes from the fact that for all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathcal{C}_{D\times D,\widetilde{X}}(\mathbf{y},\mathbf{y}) = \mathcal{C}_X(\mathbf{y}) \times \mathcal{C}_X(\mathbf{y}).$$

This ends the proof of this theorem. □

Proof of the Remark 3.3.6. Under the same as that of Theorem 3.3.3, but for $j = d$, we make the same proof as before. We get for all $\mathbf{y} \in \mathbb{R}^j$,

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathcal{C}_{D\times D-\Delta,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] \\ &= \int_{D\times D} \mathbb{E}[Y(\mathbf{x}_1)Y(\mathbf{x}_2)H(\nabla X(\mathbf{x}_1))H(\nabla X(\mathbf{x}_2)) | X(\mathbf{x}_1) = X(\mathbf{x}_2) = \mathbf{y}] \\ &\qquad \times \mathrm{p}_{X(\mathbf{x}_1),X(\mathbf{x}_2)}(\mathbf{y},\mathbf{y})\, d\mathbf{x_1}\, d\mathbf{x_2}.\end{aligned}$$

In the same form we get, for all $\mathbf{y} \in \mathbb{R}^j$ and almost surely the set A is still a differentiable manifold and since for all

$$\mathbf{y} \in \mathbb{R}^j,\ \mathcal{C}_{D\times D,\widetilde{X}}(\mathbf{y},\mathbf{y}) = \mathcal{C}_X(\mathbf{y}) \times \mathcal{C}_X(\mathbf{y}),$$

we can write recalling that in this case σ_{d-j} is the counting measure

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathcal{C}_{D\times D-\Delta,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x})\right] \\ &\quad= \mathbb{E}\left[\int_{\mathcal{C}_{D\times D,\widetilde{X}}(\mathbf{y},\mathbf{y})} \widetilde{Y}(\mathbf{x}) d\sigma_{2(d-j)}(\mathbf{x}) - \int_{\mathcal{C}_X(\mathbf{y})} Y^2(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] \\ &\quad= \mathbb{E}\left[\left(\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right)^2 - \int_{\mathcal{C}_X(\mathbf{y})} Y^2(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right],\end{aligned}$$

this ends the proof of this remark. □

Remark 3.3.9 In the same manner as in the Theorems 3.2.2 and 3.3.2, we can weaken the hypothesis $\mathbb{E}(L_X(\cdot))^{2d} < +\infty$ in the Theorem 3.3.3. More precisely, we can make the hypothesis that for almost surely $\omega \in \Omega$, $\nabla X(\omega)$ is Lipschitz or as in Remark 3.3.5, asking for the almost finiteness of
$\sup_{\mathbf{x}\in D} ||\nabla^2 X(\mathbf{x})||_{j,d}^{(s)}$. Then, it will be enough to replace conditions $\mathbf{D_i}$, $i = 1,4$ appearing in Theorem 3.3.3 by the following $\mathbf{D_i^\star}$ conditions, $i = 1,4$, conserving the hypothesis $\mathbf{H_7}$ and adding the following hypothesis $\mathbf{H_7^\star}$.

- $\mathbf{D_1^\star}$: It is the condition $\mathbf{D_1}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) \in D^4$, the density of the vector $(X(\mathbf{x_1}), X(\mathbf{x_2}), X(\mathbf{x_3}), X(\mathbf{x_4}))$ exists.

- $\mathbf{D_2^\star}$: It is the condition $\mathbf{D_2}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) \in D^4$, the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}), Z(\mathbf{x_3}), Z(\mathbf{x_4}))$ exists.

- $\mathbf{D_3^\star}$: It is the condition $\mathbf{D_3}$, plus the following hypothesis. For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) \in D^4$, the density of the vector $(Z(\mathbf{x_1}), Z(\mathbf{x_2}), Z(\mathbf{x_3}), Z(\mathbf{x_4}))$ exists.

- $\mathbf{D_4^\star}$: It is the condition $\mathbf{D_4}$, plus the following hypothesis. The function

$$(\mathbf{u_1}, \mathbf{u_2}) \longmapsto \int_{D\times D} \int_{\mathbb{R}^2\times\mathbb{R}^{2dj}} \mathbf{y_1}^2\, \mathbf{y_2}^2\, ||\dot{\mathbf{x}}_1||_{dj}^{j}\, ||\dot{\mathbf{x}}_2||_{dj}^{j}$$

$$\mathrm{p}_{Y(\mathbf{x_1}),Y(\mathbf{x_2}),X(\mathbf{x_1}),X(\mathbf{x_2}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2})}(\mathbf{y_1}, \mathbf{y_2}, \mathbf{u_1}, \mathbf{u_2}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_1)$$

$$d\dot{\mathbf{x}}_1\, d\dot{\mathbf{x}}_2\, d\mathbf{y_1}\, d\mathbf{y_2}\, d\mathbf{x_1}\, d\mathbf{x_2},$$

is a continuous function.
For almost surely $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_3, \dot{\mathbf{x}}_4) \in D^4 \times \mathbb{R}^{4dj}$ and for all $\mathbf{q} = (\mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}, \mathbf{v_2}) \in \mathbb{R}^{4j}$, the density

$$\mathrm{p}_{X(\mathbf{x_1}),X(\mathbf{x_2}),X(\mathbf{x_3}),X(\mathbf{x_4}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2}),\nabla X(\mathbf{x_3}),\nabla X(\mathbf{x_4})}(\mathbf{q}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_3, \dot{\mathbf{x}}_4),$$

of the vector

$$(X(\mathbf{x_1}), X(\mathbf{x_2}), X(\mathbf{x_3}), X(\mathbf{x_4}), \nabla X(\mathbf{x_1}), \nabla X(\mathbf{x_2}), \nabla X(\mathbf{x_3}), \nabla X(\mathbf{x_4})),$$

exists. Moreover, for all $\mathbf{y} \in \mathbb{R}^j$, the function

$$\mathbf{q} \longmapsto \int_{D^4} \int_{\mathbb{R}^{4dj}} ||\dot{\mathbf{x}}_1||_{dj}^j ||\dot{\mathbf{x}}_2||_{dj}^j ||\dot{\mathbf{x}}_3||_{dj}^j ||\dot{\mathbf{x}}_4||_{dj}^j$$
$$p_{X(\mathbf{x_1}),X(\mathbf{x_2}),X(\mathbf{x_3}),X(\mathbf{x_4}),\nabla X(\mathbf{x_1}),\nabla X(\mathbf{x_2}),\nabla X(\mathbf{x_3}),\nabla X(\mathbf{x_4})}(\mathbf{q}, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_3, \dot{\mathbf{x}}_4)$$
$$d\dot{\mathbf{x}}_1\, d\dot{\mathbf{x}}_2\, d\mathbf{x}_1\, d\mathbf{x}_2,$$

is bounded in a neighborhood of $\mathbf{q} = (\mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y})$.

Let us state the hypothesis $\mathbf{H}_7^\star$.

- $\mathbf{H}_7^\star$: For all $\mathbf{y} \in \mathbb{R}^j$, the function

$$\mathbf{q} \longmapsto \int_{D^4} p_{X(\mathbf{x_1}),X(\mathbf{x_2}),X(\mathbf{x_3}),X(\mathbf{x_4}),}(\mathbf{q}) \times$$
$$\mathbb{E}[H(\nabla X(\mathbf{x_1}))H(\nabla X(\mathbf{x_2}))H(\nabla X(\mathbf{x_3}))H(\nabla X(\mathbf{x_4}))|$$
$$(X(\mathbf{x_1}), X(\mathbf{x_2}), X(\mathbf{x_3}), X(\mathbf{x_4})) = \mathbf{q}]d\mathbf{x_1}d\mathbf{x_2}d\mathbf{x_3}d\mathbf{x_4},$$

is a bounded function in a neighborhood of $\mathbf{q} = (\mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y})$.

Chapter 4

Applications

The principal reason to have good Kac-Rice formulas is that they provide some tools to make explicitly computations that involve roots of functions as well other level functionals. Below we will present some of these applications. They can be classified by themes. First let us to mention the possibility for getting conditions under which the level functional has some moments. This is a non trivial task and it has been completely solved only in certain particular cases. Furthermore, Rice's formulas have been also applied both in physical oceanography and in the theory in dislocations of random waves propagation. Another two applications deserve to be studied: in first place the theory of random gravitational microlensings and in second place the study of the zero sets of random algebraic systems invariant under the orthogonal group, that are known in the literature as Kotlan-Shub-Smale systems. In what follows the reader will find a brief description of all of them.

4.1 Dimensions d=j=1

Classically the study of Rice's formula began with the seminal papers of Kac [18] and Rice [27]. The first one considered the number of roots of a random polynomial with standard and independent Gaussian coefficients and the second one developed formulas for studying the crossings of stationary Gaussian processes. In this subsection we will revisit these two old problems. Firstly we will give, using the formulas obtained before, necessary and sufficient conditions for the existence of both the first and the second moment of the number of crossings of a stationary Gaussian process. Secondly the research of Kac will be extended for considering random trigonometric polynomials.

4.1.1 Necessary and sufficient conditions for the first and second moment of the number of crossings

Let $X : \Omega \times \mathbb{R} \to \mathbb{R}$ be a mean zero, real and stationary Gaussian process. Let us denote its covariance function as r and μ the spectral measure assumed to be not purely discrete. Thus we have

$$r(t) = \int_{\mathbb{R}} e^{it\lambda} d\mu(\lambda).$$

The p-order spectral moment is defined as

$$\lambda_p = \int_{\mathbb{R}} \lambda^p d\mu(\lambda).$$

For $\mathbf{y} \in \mathbb{R}$ and $t > 0$ we will denote $N^X_{[0,t]}(\mathbf{y})$ the number of crossings of the level $\mathbf{y}$ by the process X on the interval $[0, t]$. We have the following theorem

Theorem 4.1.1 • *The first order Rice's formula holds if and only if* $\lambda_2 < +\infty$. *And we have for all* $\mathbf{y} \in \mathbb{R}$ *and* $t > 0$

$$\mathbb{E}[N^X_{[0,t]}(\mathbf{y})] = \frac{t}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{\mathbf{y}^2}{2\lambda_0}}.$$

- *Moreover in this case,* $\mathbb{E}[(N^X_{[0,t]}(\mathbf{y}))^2] < +\infty$ *if and only if for some* $\delta > 0$ *we have* $\frac{r''(\tau)-r''(0)}{\tau} \in \mathbb{L}^1([0,\delta], d\tau)$.

Remark 4.1.1 The first result was proved by K. Itô in [14]. In such a work the author generalizes the precedent proofs providing a definitive result. The second one is the famous Geman result [16]. He only considers the case $\mathbf{y} = \mathbf{0}$. In [20] the result was extended for all $\mathbf{y}$.

Proof of Theorem 4.1.1. Remark 3.3.1 following Theorem 3.3.1 gives the validity of the first formula whenever X belongs to $\mathbf{C}^2([0,t], \mathbb{R})$. However, the result holds with a large generality as it was shown by Itô in

[14]. For completeness we will sketch his proof. First it is proved in [14] that if $\lambda_2 < +\infty$ the process has absolutely continuous trajectories. And moreover it holds

$$N^X_{[0,t]}(\mathbf{y}) \leqslant \liminf_{\delta\to 0}\frac{1}{2\delta}\int_0^t \mathbf{1}_{\{|X(\mathbf{s})-\mathbf{y}|<\delta\}}|X'(\mathbf{s})|d\mathbf{s}.$$

Then by using Fatou's lemma and that X is Gaussian and stationary we get by denoting φ for the standard Gaussian density on $\mathbb{R}$

$$\begin{aligned}
\mathbb{E}[N^X_{[0,t]}(\mathbf{y})] &\leqslant \liminf_{\delta\to 0}\frac{t}{2\delta}\mathbb{E}[\mathbf{1}_{\{|X(\mathbf{0})-\mathbf{y}|<\delta\}}|X'(\mathbf{0})|]\\
&= \liminf_{\delta\to 0}\frac{t}{2\delta\sqrt{\lambda_0\lambda_2}}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\int_{\mathbb{R}}|\dot{\mathbf{z}}|\varphi(\frac{\mathbf{z}}{\sqrt{\lambda_0}})\varphi(\frac{\dot{\mathbf{z}}}{\sqrt{\lambda_2}})d\mathbf{z}d\dot{\mathbf{z}}\\
&= \liminf_{\delta\to 0}\frac{t}{2\delta\sqrt{\lambda_0}}\int_{\mathbf{y}-\delta}^{\mathbf{y}+\delta}\varphi(\frac{\mathbf{z}}{\sqrt{\lambda_0}})d\mathbf{z}\sqrt{\frac{2\lambda_2}{\pi}}\\
&= \frac{t}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}e^{-\frac{\mathbf{y}^2}{2\lambda_0}}.
\end{aligned}$$

Concerning the other inequality, in [14] it is proved that the following inequality holds true

$$N^X_{[0,t]}(\mathbf{y}) \geqslant \lim_{n\to+\infty}\sum_{k=1}^{2^n}\mathbf{1}_{\{(X(\frac{(k-1)t}{2^n})-\mathbf{y})(X(\frac{kt}{2^n})-\mathbf{y})<0\}}\text{ (monotonic limit).}$$

Therefore by using the monotone convergence theorem we obtain

$$\mathbb{E}[N^X_{[0,t]}(\mathbf{y})] \geqslant \lim_{n\to+\infty}2^n\mathbb{E}[\mathbf{1}_{\{(X(0)-\mathbf{y})(X(\frac{t}{2^n})-\mathbf{y})<0\}}].$$

The expectation in the right hand side can be written as

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{(X(\mathbf{0})-\mathbf{y})(X(\frac{t}{2^n})-\mathbf{y})<0\}}]\\
&= \mathbb{E}[\mathbf{1}_{(\frac{\mathbf{y}}{\sqrt{\lambda_0}},+\infty)}(\frac{X(\mathbf{0})}{\sqrt{\lambda_0}})\mathbf{1}_{(-\infty,\frac{\mathbf{y}}{\sqrt{\lambda_0}})}(\frac{X(\frac{t}{2^n})}{\sqrt{\lambda_0}})]\\
&\qquad +\mathbb{E}[\mathbf{1}_{(\frac{\mathbf{y}}{\sqrt{\lambda_0}},+\infty)}(\frac{X(\frac{t}{2^n})}{\sqrt{\lambda_0}})\mathbf{1}_{(-\infty,\frac{\mathbf{y}}{\sqrt{\lambda_0}})}(\frac{X(\mathbf{0})}{\sqrt{\lambda_0}})].
\end{aligned}$$

Then by ease of notation let us take above $\lambda_0 = 1$. Thus if Z_n stands for a Gaussian r.v. independent of $(X(\mathbf{0}), X(\frac{t}{2^n}))$ we have

$$\mathbb{E}[\mathbf{1}_{(\mathbf{y},+\infty)}(X(\mathbf{0}))\mathbf{1}_{(-\infty,\mathbf{y})}(X(\tfrac{t}{2^n}))]$$

$$= \mathbb{E}[\mathbf{1}_{(\mathbf{y},+\infty)}(X(\mathbf{0}))\mathbf{1}_{(-\infty,\mathbf{y})}(r(\frac{t}{2^n})X(\mathbf{0}) + \sqrt{1-r^2(\frac{t}{2^n})}Z_n)]$$

$$= \int_{-\infty}^{0} \varphi(\mathbf{z})d\mathbf{z} \int_{\mathbf{y}}^{\frac{\mathbf{y}-\sqrt{1-r^2(\frac{t}{2^n})}\mathbf{z}}{r(\frac{t}{2^n})}} \varphi(\mathbf{x})d\mathbf{x},$$

thus

$$2^n \int_{-\infty}^{0} \varphi(\mathbf{z})d\mathbf{z} \int_{\mathbf{y}}^{\frac{\mathbf{y}-\sqrt{1-r^2(\frac{t}{2^n})}\mathbf{z}}{r(\frac{t}{2^n})}} \varphi(\mathbf{x})d\mathbf{x} \to t\,\frac{\sqrt{\lambda_2}}{2\pi}e^{-\frac{1}{2}\mathbf{y}^2}.$$

The same process can be made for the second term. Obtaining for any λ_0 finally

$$\mathbb{E}[N^X_{[0,t]}(\mathbf{y})] \geqslant \frac{t}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}e^{-\frac{1}{2}\frac{\mathbf{y}^2}{\lambda_0}}.$$

The above procedure can be used also for proving that if $\lambda_2 = +\infty$ then $\mathbb{E}[N^X_{[0,t]}(\mathbf{y})] = +\infty$. And all the results hold in force.

For proving the second statement of the theorem we can use Remarks 3.3.5 and 3.3.6 following Theorem 3.3.3 for the case $d = j = 1$, making thus the hypothesis that X belongs to $\mathbf{C}^2([0,t],\mathbb{R})$. Thus the formula for the second factorial moment holds providing that the integral appearing in equation (4.1) is finite. Let us remark that the assumption that X possesses C^2-trajectories implies that the covariance r is C^4 and then $\lambda_4 < +\infty$. However the case $\lambda_4 = +\infty$ remains an interesting case. It is the reason why we will follow the more general way given by [12] and [16]. In [12] it is shown that

$$M_2(\mathbf{y},t) := \mathbb{E}[N^X_{[0,t]}(\mathbf{y})(N^X_{[0,t]}(\mathbf{y})-1)]$$

$$= 2\int_0^t (t-\tau)\int_{\mathbb{R}^2} |\mathbf{x}_1'||\mathbf{x}_2'|p_\tau(\mathbf{y},\mathbf{x}_1',\mathbf{y},\mathbf{x}_2')d\mathbf{x}_1'd\mathbf{x}_2'd\tau, \tag{4.1}$$

where $p_\tau(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2')$ is the density of the vector

$$(X(\mathbf{0}), X'(\mathbf{0}), X(\tau), X'(\tau)),$$

that is non-singular for $\tau > 0$, since the spectral measure μ is not purely discrete. Furthermore it is shown that if one of the term is infinity the same fact occurs for the other. In this way we will give a necessary and sufficient condition for the right hand side in the formula to be finite.

Without loss of generality we can assume that $r(\mathbf{0}) = 1$ and since $\lambda_2 < +\infty$ that r is two times differentiable.

We will begin by showing the result for the level $\mathbf{y} = 0$ that is the Geman's original result. Firstly let us write $M_2(\mathbf{y}, t)$ in another fashion by using a regression model. In first place we have

$$M_2(\mathbf{y}, t)$$

$$= 2\int_0^t (t-\tau)p_\tau(\mathbf{y}, \mathbf{y})\mathbb{E}[|X'(\mathbf{0})X'(\tau)|\,|X(\mathbf{0}) = X(\tau) = \mathbf{y}]d\tau, \qquad (4.2)$$

where $p_\tau(\mathbf{x}_1, \mathbf{x}_2)$ stands for the density of the vector $(X(\mathbf{0}), X(\tau))$. The following model will be useful

$$\begin{aligned} X'(\mathbf{0}) &= \xi + \alpha_1(\tau)X(\mathbf{0}) + \alpha_2(\tau)X(\tau) \\ X'(\tau) &= \xi^\star + \beta_1(\tau)X(\mathbf{0}) + \beta_2(\tau)X(\tau), \end{aligned}$$

where $(\xi, \xi^\star)$ is a Gaussian vector independent of $(X(\mathbf{0}), X(\tau))$, and

$$\text{Var }(\xi) = \text{ Var }(\xi^\star) := \sigma^2(\tau) = -r''(\mathbf{0}) - \frac{(r'(\tau))^2}{1-r^2(\tau)},$$

$$\rho(\tau) := \frac{\text{Cov }(\xi, \xi^\star)}{\sigma^2(\tau)} = \frac{-r''(\tau)(1-r^2(\tau)) - (r'(\tau))^2 r(\tau)}{-r''(\mathbf{0})(1-r^2(\tau)) - (r'(\tau))^2}.$$

Moreover

$$\begin{aligned} \alpha_1(\tau) = \frac{r'(\tau)r(\tau)}{1-r^2(\tau)} \quad &; \quad \alpha_2(\tau) = -\frac{r'(\tau)}{1-r^2(\tau)} \\ \beta_1(\tau) = -\alpha_2(\tau) \quad &; \quad \beta_2(\tau) = -\alpha_1(\tau). \end{aligned}$$

In this form we have

$$M_2(\mathbf{0}, t) = 2\int_0^t (t-\tau)p_\tau(\mathbf{0}, \mathbf{0})\mathbb{E}[|\xi||\xi^\star|]d\tau$$

$$= \frac{1}{\pi}\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}\mathbb{E}[|\frac{\xi}{\sigma(\tau)}||\frac{\xi^\star}{\sigma(\tau)}|]d\tau.$$

By using Cauchy-Schwarz inequality we get

$$M_2(\mathbf{0},t) \leqslant \frac{t}{\pi}\int_0^t \frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau,$$

hence if the interval of the right hand side is finite then $M_2(\mathbf{0},t) < +\infty$. But the integral is convergent if for a $\delta > 0$ we have $\int_0^\delta \frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau < +\infty$, because the integrant is continuous in $[\delta, t]$. Reciprocally denoting by a_{2k} the coefficients of the function $|x|$ in the Hermite basis of $\mathbb{L}^2(\mathbb{R}, \varphi(\mathbf{x})d\mathbf{x})$, Mehler's formula gives (see [10])

$$\begin{aligned} +\infty > M_2(\mathbf{0},t) &= 2\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}(\sum_{k=0}^{\infty} a_{2k}^2(2k)!\rho(\tau)^{2k})d\tau \\ &\geqslant 2a_0^2\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau \\ &\geqslant 2a_0^2(t-\delta)\int_0^\delta \frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau. \end{aligned}$$

We end the proof in this case if we can prove that

$$\frac{r''(\tau)-r''(0)}{\tau} \in \mathbb{L}^1([0,\delta], d\tau) \Longleftrightarrow \int_0^\delta \frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau < +\infty.$$

But this is the matter of the Lemma 4.1.1 proved below.

Now we will consider the case where $\mathbf{y}$ is any real. Let us define $m(\tau) = \dfrac{\mathbf{y}}{1+r(\tau)}\dfrac{r'(\tau)}{\sigma(\tau)}$, and let introduce the expression

$$A(m,\rho,\tau) = \mathbb{E}\big[\big|\frac{\xi}{\sigma(\tau)} - m(\tau)\big|\big|\frac{\xi^\star}{\sigma(\tau)} + m(\tau)\big|\big].$$

By using (4.2) and the regression it holds

$$M_2(\mathbf{y},t) = 2\int_0^t (t-\tau)p_\tau(\mathbf{y},\mathbf{y})\sigma^2(\tau)A(m,\rho,\tau)d\tau.$$

Applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} A(m,\rho,\tau) &\leqslant \left(\mathbb{E}\big[\frac{\xi}{\sigma(\tau)} - m(\tau)\big]^2\mathbb{E}\big[\frac{\xi^\star}{\sigma(\tau)} - m(\tau)\big]^2\right)^{\frac{1}{2}} \\ &= 1+m^2(\tau). \end{aligned} \tag{4.3}$$

Let us prove now that the function $m(\tau)$ is bounded in a neighborhood of $\tau = 0$. In this aim let us consider the asymptotic behavior of the term $\frac{r'(\tau)}{\sigma(\tau)}$. Two cases must be considered according to λ_4 is finite or not. In the former case a fourth order Taylor development of function $\frac{r'^2(\tau)}{\sigma^2(\tau)}$ gives easily that $\frac{r'(\tau)}{\sigma(\tau)} \to \frac{2\lambda_2}{\sqrt{\lambda_4-\lambda_2^2}}$. Assume now that $\lambda_4 = +\infty$. Given that $r''(\tau) - r''(0) = 2\int_0^\infty \frac{1-\cos(\tau\lambda)}{\tau}\lambda^2 d\mu(\lambda)$, we have by Fatou's lemma

$$\begin{aligned}\liminf_{\tau\to 0}\frac{r''(\tau)-r''(0))}{\tau^2} &\geqslant \int_0^{+\infty}\liminf_{\tau\to 0}\frac{1-\cos(\tau\lambda)}{\frac{(\tau\lambda)^2}{2}}\lambda^4 d\mu(\lambda)\\ &= \int_0^\infty \lambda^4 d\mu(\lambda) = +\infty.\end{aligned}$$

Moreever

$$\frac{r'^2(\tau)}{\sigma^2(\tau)} \simeq \frac{\lambda_2^3}{\frac{\lambda_2(1-r^2(\tau))-r'^2(\tau)}{\tau^4}},$$

and $\lambda_2(1-r^2(\tau)) - r'^2(\tau) = 2\lambda_2(1-r(\tau)) - r'^2(\tau) + O(\tau^4)$. Furthermore by using the l'Hospital rule

$$\lim_{\tau\to 0}\frac{2\lambda_2(1-r(\tau))-r'^2(\tau)}{\tau^4} = \lim_{\tau\to 0}\left(\frac{-r'(\tau)}{2\tau}\right)\left(\frac{r''(\tau)-r''(0))}{\tau^2}\right) = +\infty,$$

since we know that $\frac{-r'(\tau)}{2\tau} \to \frac{\lambda_2}{2}$. Thus $\frac{r'(\tau)}{\sigma(\tau)} \to 0$.
Theses computations lead us to conclude that as $\tau \to 0$, $m(\tau)$ tends to $\frac{\lambda_2 y}{\sqrt{\lambda_4-\lambda_2^2}}$ in case where $\lambda_4 < +\infty$ and to zero otherwise. In both cases we then have shown that the function $m(\tau)$ is bounded.
In this form, by using the above inequality (4.3), we readily get

$$M_2(\mathbf{y},t) \leqslant \mathbf{C}\,t\int_0^t p_\tau(\mathbf{y},\mathbf{y})\sigma^2(\tau)d\tau,$$

and by Lemma 4.1.1 this integral is finite under the Geman's condition. For proving the other implication assume that $M_2(\mathbf{y},t) < +\infty$. Thus

$$M_2(\mathbf{y},t) \geqslant 2\int_0^\delta (t-\tau)p_\tau(\mathbf{y},\mathbf{y})\sigma^2(\tau)A(m,\rho,\tau)d\tau.$$

We shall study $A(m, \rho, \tau)$.
As the function $m(\tau)$ is bounded the following expansion holds

$$|\mathbf{x} - m(\tau)| = \sum_{k=0}^{\infty} a_k(m(\tau)) H_k(\mathbf{x}),$$

where suppressing the variable τ in m, the coefficients are

$$\begin{aligned} a_0(m) &= m[2\Phi(m) - 1] + 2\varphi(m) \\ a_1(m) &= 1 - 2\Phi(m) \\ a_\ell(m) &= \frac{2}{\ell!} H_{\ell-2}(m)\varphi(m),\ \ell \geqslant 2, \end{aligned}$$

where Φ stands for the Gaussian distribution of φ.
Using that function $a_k(m)$ is even if k is even and odd otherwise, the Mehler's formula gives

$$A(m, \rho, \tau) = \sum_{k=0}^{\infty} a_k(m(\tau)) a_k(-m(\tau)) k! \rho^k(\tau)$$

$$= \sum_{k=0}^{\infty} a_{2k}^2(m(\tau))(2k)!(\rho(\tau))^{2k} - \sum_{k=0}^{\infty} a_{2k+1}^2(m(\tau))(2k+1)!(\rho(\tau))^{2k+1}.$$

But by defining the odd projection as $M_{odd}(\mathbf{x}, m) = \frac{1}{2}(|\mathbf{x} - m| - |\mathbf{x} + m|)$ it holds

$$M_{odd}(\mathbf{x}, m) = \sum_{k=0}^{\infty} a_{2k+1}(m) H_{2k+1}(\mathbf{x}).$$

Then

$$|\mathbb{E}[M_{odd}(\frac{\xi}{\sigma(\tau)}, m(\tau)) M_{odd}(\frac{\xi^\star}{\sigma(\tau)}, m(\tau))]| =$$

$$|\sum_{k=0}^{\infty} a_{2k+1}^2(m(\tau))(2k+1)!(\rho(\tau))^{2k+1}| \leqslant \mathbb{E}[M_{odd}^2(\frac{\xi}{\sigma(\tau)}, m(\tau))] =$$

$$\int_{\mathbb{R}} (\frac{1}{2}(|\mathbf{x} - m(\tau)| - |\mathbf{x} + m(\tau)|))^2 \varphi(\mathbf{x}) d\mathbf{x} \leqslant m^2(\tau).$$

Thus

$$A(m, \rho, \tau) \geqslant a_0^2(m(\tau)) - m^2(\tau).$$

Now it is easy to see that if $-m_0 \leqslant m \leqslant m_0$, then $a_0^2(m) - m^2 \geqslant \sqrt{\frac{2}{\pi}}(a_0(m_0) - m_0) > 0$.
Since function $m(\tau)$ is bounded in a neighborhood of zero, this implies that $A(m,\rho,\tau) \geqslant C$ for τ small enough. Then

$$
\begin{aligned}
+\infty &> M_2(\mathbf{y},t) \\
&\geqslant C\int_0^{\delta}(t-\tau)p_\tau(\mathbf{y},\mathbf{y})\sigma^2(\tau)d\tau \geqslant C\int_0^{\delta}\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau,
\end{aligned}
$$

and we end evoking again Lemma 4.1.1. □

Lemma 4.1.1 *There exists a $\delta > 0$ such that*

$$\frac{r''(\tau)-r''(0)}{\tau} \in \mathbb{L}^1([0,\delta],d\tau) \Longleftrightarrow \int_0^{\delta}\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau < \infty.$$

Proof of Lemma 4.1.1. Let us consider the integral

$$\int_0^{\delta}\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}}d\tau.$$

For τ small enough we have

$$\frac{\sigma^2(\tau)}{(1-r^2(\tau))^{1/2}} \simeq (\frac{1}{\lambda_2})^{\frac{3}{2}}\,\frac{-r''(\mathbf{0})(1-r^2(\tau))-(r'(\tau))^2}{\tau^3},$$

thus integrating by part

$$
\begin{aligned}
&\int_0^{\delta}\frac{-r''(\mathbf{0})(1-r^2(\tau))-(r'(\tau))^2}{\tau^3}d\tau \\
&\qquad = \frac{r''(\mathbf{0})(1-r^2(\tau))+(r'(\tau))^2}{2\tau^2}\Big|_0^{\delta} \\
&\qquad\qquad + \int_0^{\delta}\frac{r'(\tau)}{\tau}\Big(\frac{r''(\mathbf{0})r(\tau)-r''(\tau)}{\tau}\Big)d\tau \\
&\qquad = \frac{r''(\mathbf{0})(1-r^2(\delta))+(r'(\delta))^2}{2\delta^2} \\
&\qquad\qquad + \int_0^{\delta}\frac{r'(\tau)}{\tau}\,r''(\mathbf{0})\left(\frac{r(\tau)-1}{\tau}\right)d\tau \\
&\qquad\qquad + \int_0^{\delta}\frac{-r'(\tau)}{\tau}\left(\frac{r''(\tau)-r''(\mathbf{0})}{\tau}\right)d\tau,
\end{aligned}
$$

since $\frac{r'(\tau)}{\tau}\left(\frac{r(\tau)-1}{\tau}\right) \simeq \lambda_2^2 \frac{\tau}{2} \in L^1([0,\delta], d\tau)$.
Finally and since $\frac{-r'(\tau)}{\tau} \to \lambda_2$, the above integral is finite if and only if

$$\int_0^\delta \frac{r''(\tau) - r''(\mathbf{0})}{\tau} d\tau < +\infty.$$

□

4.1.2 Numbers of roots of random trigonometric polynomials

In the sequel we will study the asymptotic behavior of the random Gaussian trigonometric polynomials. For any $N \in \mathbb{N}^\star$ and for two independent sequences of i.i.d standard Gaussian random variables $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ these functions are defined as

$$X_N(\mathbf{t}) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (a_n \sin n\mathbf{t} + b_n \cos n\mathbf{t}).$$

The number of zeros of such a process have been extensively study in the recent times (see [17] for example). Process X_N is a mean zero infinitely differentiable stationary Gaussian process. We can define as before $N_{[0,2\pi)}^{X_N}(\mathbf{y})$ as the number of crossings of level $\mathbf{y}$ of these trigonometric polynomials on the time interval $[0, 2\pi)$. The smoothness of these polynomials implies that the Rice's formula holds. The ingredients needed for its application are

$$\mathbb{E}[X_N^2(\mathbf{0})] = 1 \qquad \mathbb{E}[(X_N'(\mathbf{0}))^2] = \frac{1}{N}\sum_{n=1}^{N} n^2 = \frac{(N+1)(2N+1)}{6}.$$

Hence

$$\mathbb{E}[N_{[0,2\pi)}^{X_N}(\mathbf{y})] = 2\pi\sqrt{E[(X_N'(\mathbf{0}))^2]}\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{\mathbf{y}^2}{2}}}{\sqrt{2\pi}}$$

$$= \frac{2}{\sqrt{3}}\sqrt{\frac{(N+1)(2N+1)}{2}}e^{-\frac{\mathbf{y}^2}{2}}.$$

Yielding

$$\lim_{N\to\infty} \frac{\mathbb{E}[N^{X_N}_{[0,2\pi)}(\mathbf{y})]}{N} = \frac{2}{\sqrt{3}} e^{-\frac{y^2}{2}}.$$

For computing the variance and its asymptotic value we need to consider the rescaled process: $Y_N(\mathbf{t}) = X_N(\frac{\mathbf{t}}{N})$. Given that the covariance function X_N is

$$r_{X_N}(\mathbf{t}) = \frac{1}{N}\sum_{n=1}^{N} \cos n\mathbf{t} = \frac{1}{N}\cos(\frac{(N+1)\mathbf{t}}{2})\frac{\sin(\frac{N\mathbf{t}}{2})}{\sin\frac{\mathbf{t}}{2}},$$

we get

$$r_{Y_N}(\mathbf{t}) \to r_X(\mathbf{t}) := \frac{\sin \mathbf{t}}{\mathbf{t}}.$$

Similar results can be obtained for the first and second derivative of r_{Y_N}. The above result leads us to consider the Sine cardinal process which has as covariance the function r_X. In [4] was proved that by constructing the processes Y_N and X in the same probability space and if we define

$$B_N$$
$$= \mathbb{E}\{\big[([N^{X_N}_{[0,2\pi)}(\mathbf{0}) - \mathbb{E}[N^{X_N}_{[0,2\pi)}(\mathbf{0})]) - ([N^{X}_{[0,2\pi N)}(\mathbf{0}) - \mathbb{E}[N^{X}_{[0,2\pi N)}(\mathbf{0})])\big]^2\},$$

it holds that $\frac{B_N}{N} \to 0$. This result entails that

$$\lim_{N\to+\infty} \frac{1}{N}\mathrm{Var}(N^{X_N}_{[0,2\pi)}(\mathbf{0})) = \lim_{N\to+\infty}\frac{1}{N}\mathrm{Var}(N^{X}_{[0,2\pi N)}(\mathbf{0})),$$

and the latter quantity is

$$= \frac{2}{\sqrt{3}} + 2\int_0^\infty [\frac{\mathbb{E}[|X'(\mathbf{0})X'(\tau)|\,|X(\mathbf{0}) = X(\tau) = \mathbf{0}]}{\sqrt{1-(\frac{\sin\tau}{\tau})^2}} - \frac{1}{3\pi}]d\tau.$$

4.2 Dimensions d>1

4.2.1 Sea modeling applications

In this subsection our results will be used to give some theoretical justifications to the work of Podgórski & Rychlik [26]. This article provides several applications to random sea waves. Like those authors let us consider two random fields

$$X : \mathbb{R}^d \to \mathbb{R}(\text{ with } d > j = 1) \text{ and } V : \mathbb{R}^d \to \mathbb{R}^{d+1}.$$

This last field is defined as $V(\mathbf{x}) = (X(\mathbf{x}), \nabla X(\mathbf{x}))$. It is the argument of the function G in (3.18) but suppressing the explicitly dependence on $\mathbf{x}$ and also on the field W. That is $Y(\mathbf{x}) = G(X(\mathbf{x}), \nabla X(\mathbf{x}))$.

Moreover, for sea applications either the field X is Gaussian and models the sea surface or it is the envelope field (defined below).

We will discuss first the case where X belongs to $\mathbf{C}^2(D, \mathbb{R})$, with mean zero and is a stationary Gaussian field with $\sigma^2 = \text{Var}(X(\mathbf{0})) = \mathbb{E}[X^2(\mathbf{0})] > 0$. Then Remark 3.3.1 following Theorem 3.3.1 applies and we have for all $\mathbf{y} \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-1}(\mathbf{x})\right] \\
&\qquad = \int_D \mathrm{p}_{X(\mathbf{x})}(\mathbf{y})\, \mathbb{E}\left[Y(\mathbf{x}) \left\|\nabla X(\mathbf{x})\right\|_d \left| X(\mathbf{x}) = \mathbf{y}\right.\right] d\mathbf{x} \\
&\qquad = \sigma_d(D) \mathbb{E}\left[Y(\mathbf{0}) \left\|\nabla X(\mathbf{0})\right\|_d \left| X(\mathbf{0}) = \mathbf{y}\right.\right] \frac{e^{-\frac{1}{2\sigma^2}\mathbf{y}^2}}{(2\pi)^{\frac{1}{2}}\sigma}.
\end{aligned}$$

The following notion is also introduced in [26]. One defines the distribution of V over the level set by taking first $G(\mathbf{z}) = \mathbf{1}_A(\mathbf{z})$ for $\mathbf{z} \in \mathbb{R}^{d+1}$ and A a Borel set of $\mathbb{R}^{d+1}$. We have $Y(\mathbf{x}) = \mathbf{1}_A(V(\mathbf{x}))$ and setting

$$\begin{aligned}
\mathbb{P}\{V(\mathbf{x}) \in A \left| X(\mathbf{x}) = \mathbf{y}\right.\} &:= \frac{\mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} \mathbf{1}_A(V(\mathbf{x})) d\sigma_{d-1}(\mathbf{x})\right]}{\mathbb{E}\left[\int_{\mathcal{C}_X(\mathbf{y})} d\sigma_{d-1}(\mathbf{x})\right]} \\
&= \frac{\mathbb{E}\left[\mathbf{1}_A(V(\mathbf{0})) \left\|\nabla X(\mathbf{0})\right\|_d \left| X(\mathbf{0}) = \mathbf{y}\right.\right]}{\mathbb{E}\left[\left\|\nabla X(\mathbf{0})\right\|_d\right]}. \\
&= \frac{\mathbb{E}\left[\mathbf{1}_A(\mathbf{y}, \nabla X(\mathbf{0})) \left\|\nabla X(\mathbf{0})\right\|_d\right]}{\mathbb{E}\left[\left\|\nabla X(\mathbf{0})\right\|_d\right]} \qquad (4.4)
\end{aligned}$$

For applying the formula to sea waves modeling we set $d = 3$. Let us use the notation of sea modeling taken from [26]. We have a mean zero and stationary Gaussian field $X(t, \mathbf{p}) := \zeta(t, x, y)$, $\mathbf{p} = (x, y)$, that models the sea surface. For introducing it let $M(\lambda_1, \lambda_2, \omega)$ be a random spectral Gaussian measure, restricted to the airy manifold $\Lambda = \{\left\|\overrightarrow{\mathbf{k}}\right\|_2^2 = \frac{\omega^4}{g^2}\}$ where $\overrightarrow{\mathbf{k}} = (\lambda_1, \lambda_2)$. One defines

$$\zeta(t, x, y) = \int_\Lambda e^{i(\lambda_1 x + \lambda_2 y + \omega t)} dM(\lambda_1, \lambda_2, \omega).$$

In this manner restricting the stochastic integral to the set

$$\Lambda^+ = \{(\lambda_1, \lambda_2, \omega) : \omega \geqslant 0, \left\|\overrightarrow{\mathbf{k}}\right\|_2 = \frac{\omega^2}{g}\},$$

by using polar coordinates we can write

$$\zeta(t, x, y)$$
$$= 2\int_0^\infty \int_{-\pi}^{\pi} \cos(\left\|\overrightarrow{\mathbf{k}}\right\|_2 \cos(\theta)\, x + \left\|\overrightarrow{\mathbf{k}}\right\|_2 \sin(\theta)\, y + \omega t) d\mathrm{c}(\omega, \theta),$$

where c is an aleatory measure with independent Gaussian increments. The covariance function results

$$\Gamma(t, \mathbf{p}) := \mathbb{E}[\zeta(0,0,0)\zeta(t, x, y)]$$
$$= 2\int_0^\infty \int_{-\pi}^{\pi} \cos(\left\|\overrightarrow{\mathbf{k}}\right\|_2 \cos(\theta)\, x + \left\|\overrightarrow{\mathbf{k}}\right\|_2 \sin(\theta)\, y + \omega t) S(\omega, \theta) d\omega d\theta,$$

here function $2S(\cdot, \cdot)$ is the physical spectral density.

For establishing the next results it will be necessary to give a digression about ergodic theory. The following text has been taken from [5]. For a given subset $D \subset \mathbb{R}^2$ and for each $t > 0$, let us define $\mathcal{A}_t = \sigma\{X(\tau, \mathbf{p}) : \tau > t, \mathbf{p} \in D\}$ and consider the σ-algebra of t-invariant sets $\mathcal{A} = \bigcap_t \mathcal{A}_t$. Moreover, we assume that for all $\mathbf{p} \in D$ it holds $\Gamma(t, \mathbf{p}) \to_{t\to\infty} 0$. It is well known that under this condition, the σ-algebra $\mathcal{A}$ is trivial, that is, it only contains events having probability zero or one (see e.g. [12] Chapter 7).

Now for each $t > 0$ and $\mathbf{y} \in \mathbb{R}$ we define the level set

$$\mathcal{C}_D^\zeta(t, \mathbf{y}) = \{\mathbf{p} \in D : \zeta(t, \mathbf{p}) = \mathbf{y}\}$$

and the following functional

$$\mathcal{Z}(t) = \int_{\mathcal{C}_D^{\zeta}(t,\mathbf{y})} Y(t,\mathbf{p})d\sigma_1(\mathbf{p}).$$

Furthermore, in the sequel we assume that

$$Y(t,\mathbf{p}) = G(\zeta(t,\mathbf{p}), \nabla_{\mathbf{p}}\zeta(t,\mathbf{p})),$$

where $\nabla_{\mathbf{p}}$ is the gradient operator with respect to the space variables x, y. The process $\{\mathcal{Z}(t) : t \in \mathbb{R}^+\}$ is strictly stationary and has a finite mean and is Riemann-integrable. The ergodic theorem gives that almost surely as T tends to infinity

$$\frac{1}{T}\int_0^T \mathcal{Z}(t)dt \to \mathbb{E}_{\mathcal{B}}[Z(\mathbf{0})],$$

where $\mathcal{B}$ is the σ-algebra of t-invariant events associated to the process $\mathcal{Z}(\mathbf{t})$. Since for each t, $\mathcal{Z}(t)$ is $\mathcal{A}_t$- measurable, it follows that $\mathcal{B} \subset \mathcal{A}$ so that $\mathbb{E}_{\mathcal{B}}[\mathcal{Z}(\mathbf{0})] = \mathbb{E}[\mathcal{Z}(\mathbf{0})]$. Thus

$$\mathbb{E}_{\mathcal{B}}[\mathcal{Z}(\mathbf{0})] = \mathbb{E}[\int_{\mathcal{C}_D^{\zeta}(0,\mathbf{y})} Y(0,\mathbf{p})d\sigma_1(\mathbf{p})]$$

$$= \sigma_2(D)\mathbb{E}\left[Y(0,\mathbf{0})\left\|\nabla_{\mathbf{p}}\zeta(0,\mathbf{0})\right\|_2 |\zeta(0,\mathbf{0}) = \mathbf{y}\right]\frac{e^{-\frac{\mathbf{y}^2}{2\lambda_{000}}}}{(2\pi\lambda_{000})^{\frac{1}{2}}},$$

where $\lambda_{000} = \mathbb{E}[\zeta^2(0,\mathbf{0})]$. The above formula can be used to get the distribution of velocities as defined in (4.4) (cf. [5]).

Next we will consider the case where the observed field, denoted as $E(t,\mathbf{p})$, is the envelope field of $X(t,\mathbf{p})$. First let define the Hilbert transform of ζ as the Gaussian field

$$\hat{\zeta}(t,x,y)$$
$$= 2\int_0^{\infty}\int_{-\pi}^{\pi} \sin(\left\|\overrightarrow{\mathbf{k}}\right\|_2 \cos(\theta)\, x + \left\|\overrightarrow{\mathbf{k}}\right\|_2 \sin(\theta)\, y + \omega t)dc(\omega,\theta).$$

The real envelope $E(t,x,y)$ is defined as

$$E(t,\mathbf{p}) = \sqrt{\zeta^2(t,x,y) + \hat{\zeta}^2(t,x,y)}. \tag{4.5}$$

We can write the process E in the following form. Set

$$Z(t,\mathbf{p}) = (\zeta(t,x,y), \hat{\zeta}(t,x,y)).$$

Then if $F(\mathbf{z}) = \|\mathbf{z}\|_2$, it holds $E(t,\mathbf{p}) = F(Z(t,\mathbf{p}))$. The function F satisfies condition $\mathbf{B_3}$ except for $\mathbf{z} = 0$, but this does not matter because $\mathbb{P}\{Z(0,\mathbf{0}) = 0\} = 0$. Furthermore a straightforward calculation shows that the process X verifies assumption $(\mathbf{S})$ in Proposition 3.3.2. We can then apply Remark 3.3.1 following Theorem 3.3.1 with condition $\mathbf{B_3}$ and hypothesis $(\mathbf{S})$ replacing condition $\mathbf{C_3}$.

Moreover, the density of $E(0,\mathbf{0})$, in the point $\mathbf{y} > 0$, is the Rayleigh density $\frac{1}{\sigma_\zeta^2}\mathbf{y}e^{-\frac{\mathbf{y}^2}{2\sigma_\zeta^2}}$, that exists and is continuous if $\sigma_\zeta^2 = Var(\zeta(0,\mathbf{0})) > 0$.

For each $t > 0$ and $\mathbf{y} > 0$ we define the level set

$$\mathcal{C}_D^E(t,\mathbf{y}) = \{\mathbf{p} \in D : E(t,\mathbf{p}) = \mathbf{y}\},$$

and the functional

$$\mathcal{Z}_E^Y(t) = \int_{\mathcal{C}_D^E(t,\mathbf{y})} Y(t,\mathbf{p})d\sigma_1(\mathbf{p}).$$

Invoking again the ergodic theorem we get almost surely as T tends to infinity

$$\frac{\int_0^T Z_E^Y(t)dt}{\int_0^T Z_E^1(t)dt} \to \frac{\mathbb{E}[Y(0,\mathbf{0})\,\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2\,|E(0,\mathbf{0}) = \mathbf{y}]}{\mathbb{E}[\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2\,|E(0,\mathbf{0}) = \mathbf{y}]}, \tag{4.6}$$

but

$$\mathbb{E}[\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2\,|E(0,\mathbf{0})\mathbf{y}]$$

$$= \frac{1}{\mathbf{y}}\mathbb{E}[\|\zeta(0,\mathbf{0})\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) + \hat{\zeta}(0,\mathbf{0})\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0})\|_2\,|E(0,\mathbf{0}) = \mathbf{y}],$$

thus conditioning and defining

$$f_\pm(\mathbf{y},z_1) = \mathbb{E}[\left\|z_1\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) \pm \sqrt{\mathbf{y}^2 - z_1^2}\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0})\right\|_2]$$

we get

$$\mathbb{E}[\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2 \,|E(0,\mathbf{0}) = \mathbf{y}]$$
$$= \frac{1}{\mathbf{y}}\int_{-\mathbf{y}}^{\mathbf{y}} (f_+(\mathbf{y},\mathbf{z}_1) + f_-(\mathbf{y},\mathbf{z}_1))p_{\zeta(0,\mathbf{0})}(\mathbf{z}_1)d\mathbf{z}_1,$$

and given that $\hat{\zeta}$ has the same distribution of $-\hat{\zeta}$ we finally have

$$\mathbb{E}[\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2 \,|E(0,\mathbf{0}) = \mathbf{y}] = \frac{2}{\mathbf{y}}\int_{-\mathbf{y}}^{\mathbf{y}} f_+(\mathbf{y},\mathbf{z}_1)p_{\zeta(0,\mathbf{0})}(\mathbf{z}_1)d\mathbf{z}_1.$$

Let us consider the numerator in the right hand side of the expression (4.6),

$$\mathbb{E}[Y(0,\mathbf{0}) \,\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2 \,|E(0,\mathbf{0}) = \mathbf{y}]$$
$$= \frac{1}{\mathbf{y}}\mathbb{E}[G(\mathbf{y}, \frac{1}{\mathbf{y}}(\zeta(0,\mathbf{0})\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) + \hat{\zeta}(0,\mathbf{0})\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0})))$$
$$\times||\zeta(0,\mathbf{0})\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) + \hat{\zeta}(0,\mathbf{0})\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0})||_2 \,|E(0,\mathbf{0}) = \mathbf{y}].$$

The same argument as above yields

$$\mathbb{E}[Y(0,\mathbf{0}) \,\|\nabla_{\mathbf{p}}E(0,\mathbf{0})\|_2 \,|E(0,\mathbf{0}) = \mathbf{y}]$$
$$= \frac{2}{\mathbf{y}}\int_{-\mathbf{y}}^{\mathbf{y}} \mathcal{F}_+(\mathbf{y},\mathbf{z}_1)p_{\zeta(0,\mathbf{0})}(\mathbf{z}_1)d\mathbf{z}_1,$$

where

$$\mathcal{F}_+(\mathbf{y},\mathbf{z}_1) = E[G(\mathbf{y}, \frac{1}{\mathbf{y}}(\mathbf{z}_1\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) + \sqrt{\mathbf{y}^2 - \mathbf{z}_1^2}\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0}))) \times$$
$$\left\|\mathbf{z}_1\nabla_{\mathbf{p}}\zeta(0,\mathbf{0}) + \sqrt{\mathbf{y}^2 - \mathbf{z}_1^2}\nabla_{\mathbf{p}}\hat{\zeta}(0,\mathbf{0}))\right\|_2].$$

Hence the term in the right hand side of (4.6) becomes equal to

$$\frac{\int_{-\mathbf{y}}^{\mathbf{y}} \mathcal{F}_+(\mathbf{y},\mathbf{z}_1)p_{\zeta(0,\mathbf{0})}(\mathbf{z}_1)d\mathbf{z}_1}{\int_{-\mathbf{y}}^{\mathbf{y}} f_+(\mathbf{y},\mathbf{z}_1)p_{\zeta(0,\mathbf{0})}(\mathbf{z}_1)d\mathbf{z}_1}.$$

In certain important cases this terms can be computed explicitly using only the spectral moments of the processes ζ, $\hat{\zeta}$, $\nabla_{\mathbf{p}}\zeta$, $\nabla_{\mathbf{p}}\hat{\zeta}$.

4.2.2 Berry and Dennis dislocations

In this part of the work we will give an outline about the applications of the Rice's formulae to some notions in physics that are known as: dislocations of random waves. The motivation for such a study is the seminal paper of Berry & Dennis [9], where based on physical grounds several novelty notions were introduced.

We will consider two independent mean zero isotropic Gaussian random fields belonging to $\mathbf{C}^2(D,\mathbb{R})$, ξ, η : $\Omega \times \mathbb{R}^2 \to \mathbb{R}$, defined trough their spectral representation

$$\xi(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(<\mathbf{x},\mathbf{k}>)\left(\frac{\Pi(k)}{k}\right)^{\frac{1}{2}} dW_1(\mathbf{k}) - \int_{\mathbb{R}^2} \sin(<\mathbf{x},\mathbf{k}>)\left(\frac{\Pi(k)}{k}\right)^{\frac{1}{2}} dW_2(\mathbf{k})$$

$$\eta(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(<\mathbf{x},\mathbf{k}>)\left(\frac{\Pi(k)}{k}\right)^{\frac{1}{2}} dW_2(\mathbf{k}) + \int_{\mathbb{R}^2} \sin(<\mathbf{x},\mathbf{k}>)\left(\frac{\Pi(k)}{k}\right)^{\frac{1}{2}} dW_1(\mathbf{k}),$$

where $< \cdot,\cdot >$ stands for the scalar product in $\mathbb{R}^2$ and $\mathbf{k} = (k_1,k_2)$, $k = \|\mathbf{k}\|_2$, $\Pi(k)$ is the isotropic spectral density and $\mathbf{W} = W_1 + iW_2$ is a standard complex orthogonal Gaussian measure on $\mathbb{R}^2$. Without loss of generality, we may assume that $E[\xi(\mathbf{0})]^2 = E[\eta(\mathbf{0})]^2 = 1$.
Defining the complex wave $\psi(\mathbf{x}) = \xi(\mathbf{x}) + i\eta(\mathbf{x})$ the dislocations are the ensemble of zeros of ψ. That is

$$N_D^{\psi}(\mathbf{0}) = \#\{\mathbf{x} : \psi(\mathbf{x}) = \mathbf{0}\} = \#\{\mathbf{x} : \xi(\mathbf{x}) = \eta(\mathbf{x}) = \mathbf{0}\}.$$

In [9] (see formulas (2.7) and (4.6)) it is defined the expected number by unit of area of dislocation points as

$$d_2 = \frac{\mathbb{E}[\#\{\mathbf{x} \in D : \psi(\mathbf{x}) = \mathbf{0}\}]}{\sigma_2(D)}$$

$$= \frac{\lambda_2}{(2\pi)^2}\mathbb{E}[|\frac{\xi_x(\mathbf{0})}{\sqrt{\lambda_2}}\frac{\eta_y(\mathbf{0})}{\sqrt{\lambda_2}} - \frac{\xi_y(\mathbf{0})}{\sqrt{\lambda_2}}\frac{\eta_x(\mathbf{0})}{\sqrt{\lambda_2}}|] = \frac{\lambda_2}{2\pi},$$

where ξ_x, ξ_y, η_x and η_y stand for the derivatives of first order of ξ and η and $\lambda_2 = \mathbb{E}[\xi_x(\mathbf{0})]^2 = \mathbb{E}[\xi_y(\mathbf{0})]^2 = \mathbb{E}[\eta_x(\mathbf{0})]^2 = \mathbb{E}[\eta_y(\mathbf{0})]^2$.
And here we will study also the length of the set of zeros of each coor-

dinate process (length of nodal curves)

$$\sigma_1(C_\xi(\mathbf{0})) \stackrel{d}{=} \sigma_1(C_\eta(\mathbf{0})).$$

Thus we have the definition of length of nodal curves for unity of area

$$\mathcal{L} = \frac{\mathbb{E}[\sigma_1(C_\xi(\mathbf{0}))]}{\sigma_2(D)} = \frac{\mathbb{E}[\sigma_1(C_\eta(\mathbf{0}))]}{\sigma_2(D)}.$$

In the cited work of Berry & Dennis other notions have been defined related to the following two integrals

$$\int_{\{\mathbf{x}\in D:\, \psi(\mathbf{x})=\mathbf{0}\}} Y(\mathbf{x}) d\sigma_0(\mathbf{x}) = \sum_{\mathbf{x}\in\{\mathbf{x}\in D:\, \psi(\mathbf{x})=\mathbf{0}\}} Y(\mathbf{x}) \text{ and } \int_{C_\xi(\mathbf{0})} Y(\mathbf{x}) d\sigma_1(\mathbf{x}),$$

for the first integral we must recall that σ_0 is the counting measure. For instance in [9] it is introduced the dislocation curvature. In the sequel we will consider instead the curvature of one of the nodal curve, defined using for example ξ. The curvature of the nodal curve $\xi(\mathbf{x}) = \xi(x,y) = \mathbf{0}$, is the quantity

$$\kappa(\mathbf{x}) = \frac{|(\xi_{xx}(\mathbf{x})\xi_x^2(\mathbf{x}) - 2\xi_{xy}(\mathbf{x})\xi_x(\mathbf{x})\xi_y(\mathbf{x}) + \xi_{yy}(\mathbf{x})\xi_y^2(\mathbf{x}))|}{\|\nabla\xi\|_2^3}.$$

For the interval $[0,\kappa_1]$ by defining $Y(\mathbf{x}) = \mathbf{1}_{[0,\kappa_1]}(\kappa(\mathbf{x}))$ we have a particular case of a function $Y(\mathbf{x}) = G(\nabla\xi(\mathbf{x}), \nabla^2\xi(\mathbf{x}))$, where the operator ∇^2 denotes the second order differential. For these functions similarly that in Theorem 3.3.1, it can be proven a Rice's formula obtaining

$$\begin{aligned}
&\mathbb{E}[\int_{C_\xi(\mathbf{0})} 1_{[0,\kappa_1]}(\kappa(\mathbf{x}))d\sigma_1(\mathbf{x})] \\
&\qquad = \sigma_2(D)\mathbb{E}[1_{[0,\kappa_1]}(\kappa(\mathbf{0}))\, \|\nabla\xi(\mathbf{0})\|_2 \mid \xi(\mathbf{0}) = \mathbf{0}]p_{\xi(\mathbf{0})}(\mathbf{0}) \\
&\qquad = \frac{\sigma_2(D)}{\sqrt{2\pi}} E[1_{[0,\kappa_1]}(\kappa(\mathbf{0}))\, \|\nabla\xi(\mathbf{0})\|_2 \mid \xi(\mathbf{0}) = \mathbf{0}].
\end{aligned}$$

The independence between $\nabla\xi(\mathbf{0})$ and $(\xi(\mathbf{0}), \nabla^2\xi(\mathbf{0}))$ allows writing a regression model that simplifies the last expression. Moreover,

$$\mathbb{E}[\sigma_1(C_\xi(\mathbf{0}))] = \tfrac{\sqrt{\lambda_2}\sigma_2(D)}{\sqrt{2\pi}}\mathbb{E}[\sqrt{\tfrac{\xi_x^2(\mathbf{0})}{\lambda_2}+\tfrac{\xi_y^2(\mathbf{0})}{\lambda_2}}]$$

$$= \frac{\sqrt{\lambda_2}\sigma_2(D)}{\sqrt{2\pi}}\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\rho^2 e^{-\frac{1}{2}\rho^2}d\theta d\rho = \frac{\sqrt{\lambda_2}\sigma_2(D)}{2},$$

(as a bonus we get $\mathcal{L} = \frac{\sqrt{\lambda_2}}{2}$).

Furthermore, in order to obtain an interpretation for the distribution of κ over the level set of ξ we take the ratio of the two last expectations obtaining

$$\frac{\mathbb{E}[\int_{C_\xi(\mathbf{0})} 1_{[0,\kappa_1]}(\kappa(\mathbf{x}))d\sigma_1(\mathbf{x})]}{\mathbb{E}[\sigma_1(C_\xi(\mathbf{0}))]}$$

$$= \frac{1}{\sqrt{\lambda_2}}\sqrt{\frac{2}{\pi}}\mathbb{E}[1_{[0,\kappa_1]}(\kappa(\mathbf{0}))\,\|\nabla\xi(\mathbf{0})\|_2 \mid \xi(\mathbf{0}) = \mathbf{0}].$$

Using the independence it can be written as

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^{2\pi}\mathbb{E}[\mathbf{1}_{[0,\rho\sqrt{\lambda_2}\kappa_1]}(|\xi_{xx}(\mathbf{0})\cos^2\theta - 2\xi_{xy}(\mathbf{0})\cos\theta\sin\theta$$
$$+\xi_{yy}(\mathbf{0})\sin^2\theta|)\,|\xi(\mathbf{0}) = \mathbf{0}] \times \rho^2\frac{e^{-\frac{\rho^2}{2}}}{2\pi}d\rho d\theta. \quad (4.7)$$

A regression model yields that the following relation holds true

$$[\xi_{xx}(\mathbf{0})\cos^2\theta - 2\xi_{xy}(\mathbf{0})\cos\theta\sin\theta + \xi_{yy}(\mathbf{0})\sin^2\theta|\,\xi(\mathbf{0}) = \mathbf{0}] \stackrel{d}{=}$$
$$N(0,\sigma^2(\theta,\lambda_4,\lambda_{22},\lambda_2)),$$

where $\lambda_4 = \mathbb{E}[\xi_{xx}^2(\mathbf{0})] = \mathbb{E}[\xi_{yy}^2(\mathbf{0})]$,

$$\lambda_{22} = \mathbb{E}[\xi_{xy}^2(\mathbf{0})] \text{ and } -\lambda_2 = \mathbb{E}[\xi(\mathbf{0})\xi_{xx}(\mathbf{0})] = \mathbb{E}[\xi(\mathbf{0})\xi_{yy}(\mathbf{0})].$$

Then the expression (4.7) is equal to

$$= \tfrac{1}{2\pi^2}\int_0^\infty\int_0^{2\pi}\int_{-\kappa_1\sqrt{\lambda_2}\rho/\sigma(\theta,\lambda_4,\lambda_{22},\lambda_2)}^{\kappa_1\sqrt{\lambda_2}\rho/\sigma(\theta,\lambda_4,\lambda_{22},\lambda_2)}\rho^2 e^{-\frac{\rho^2}{2}}e^{-\frac{\mathbf{u}^2}{2}}d\rho d\theta d\mathbf{u}$$

$$= \frac{1}{\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{\kappa_1\sqrt{\lambda_2}\rho/\sigma(\theta,\lambda_4,\lambda_{22},\lambda_2)}\rho^2 e^{-\frac{\rho^2}{2}}e^{-\frac{\mathbf{u}^2}{2}}d\rho d\theta d\mathbf{u} := \mathcal{K}(\kappa_1).$$

The density of this distribution is

$$\frac{d}{d\kappa_1}\mathcal{K}(\kappa_1)$$

$$= \frac{\sqrt{\lambda_2}}{\pi^2}\int_0^\infty\int_0^{2\pi}\rho^3/\sigma(\theta,\lambda_4,\lambda_{22},\lambda_2)e^{-\frac{\rho^2}{2}}e^{-\frac{(\kappa_1\sqrt{\lambda_2}\rho/\sigma(\theta,\lambda_4,\lambda_{22},\lambda_2))^2}{2}}d\rho d\theta$$

$$= \frac{\sqrt{\lambda_2}}{\pi^2}\int_0^\infty\int_0^{2\pi}\frac{\sigma^3(\theta,\lambda_4,\lambda_{22},\lambda_2)}{(\sigma^2(\theta,\lambda_4,\lambda_{22},\lambda_2)+\kappa_1^2\lambda_2)^2}\mathbf{v}^3e^{-\frac{1}{2}\mathbf{v}^2}d\mathbf{v}d\theta$$

$$= \frac{2\sqrt{\lambda_2}}{\pi^2}\int_0^{2\pi}\frac{\sigma^3(\theta,\lambda_4,\lambda_{22},\lambda_2)}{(\sigma^2(\theta,\lambda_4,\lambda_{22},\lambda_2)+\kappa_1^2\lambda_2)^2}d\theta.$$

The last part of this subsection is aimed to compute some second order Rice's formulas. Let first introduce the dislocation correlation at distance R defined as $g(R)$ in [9]. For defining this quantity we first consider the second factorial moment of the random variable $N_D^{\psi}(\mathbf{0})$ that is

$$\mathbb{E}[\#\{\mathbf{x}\in D:\ \psi(\mathbf{x})=\mathbf{0}\}(\#\{\mathbf{x}\in D:\ \psi(\mathbf{x})=\mathbf{0}\}-1)]$$

$$=\int\int_{D\times D}A(\mathbf{x}_1,\mathbf{x}_2)d\mathbf{x}_1d\mathbf{x}_2,$$

where by using the Rice's formula it holds

$$A(\mathbf{x}_1,\mathbf{x}_2)=\mathbb{E}[|\det\nabla\psi(\mathbf{x}_1)||\det\nabla\psi(\mathbf{x}_2)|\,|\psi(\mathbf{x}_1)=\psi(\mathbf{x}_2)=\mathbf{0}].$$

By using the invariance with respect to rotations and translations it holds

$$A(\mathbf{x}_1,\mathbf{x}_2)=A((0,0),(\|\mathbf{x}_1-\mathbf{x}_2\|_2,0)):=A((0,0),(R,0)):=g(R).$$

In the last equality we have set $\|\mathbf{x}_1-\mathbf{x}_2\|_2=R$.

Moreover, by dropping the absolute value of the determinant of the Jacobian of ψ we can introduce

$$B(\mathbf{x}_1,\mathbf{x}_2)=\mathbb{E}[\det\nabla\psi(\mathbf{x}_1)\det\nabla\psi(\mathbf{x}_2)\,|\psi(\mathbf{x}_1)=\psi(\mathbf{x}_2)=\mathbf{0}].$$

Thus the *charge correlation function* (cf. [9]) is defined as

$$g_Q(R):=B((0,0),(R,0)).$$

Both in [9] and [5] a closed elementary expression for $g(R)$ was obtained by using an expression for the absolute value function as a Fourier integral. Nevertheless, the computation is not a trivial one. The interested reader can consult these works. On the other hand, the function $g_Q(R)$ can be written as the conditional expectation of a sum of products of four standard Gaussian random variables. As instance let us consider the first term. That is

$$\mathbb{E}[\xi_x(0,0)\eta_y(0,0)\xi_x(R,0)\eta_y(R,0) \,|\psi(0,0) = \psi(R,0)) = \mathbf{0}],$$

then we make the regression of the random variables representing the derivatives with respect to the vector $(\psi(0,0), \psi(R,0))$. An elementary Gaussian computation gives the result.

4.2.3 Gravitational stochastic microlensing

In this part we only sketch an application to gravitational cosmology. The main reason to present it is that the used Rice's formula is shown for a non-Gaussian process. However, this subsection is built more as an ilustration than a true formal mathematical development.

We must point out that all the matter corresponding to this subsection comes from the article of Peters et al. [25]. Moreover, for the background this work needed to be complemented with the book [24].

Let $\{\xi_i\}$ be g independent and identically distributed random variables uniformly distributed over the disc of radius R in $\mathbb{R}^2$. They will be considered as the positions of the stars. The following random field can be defined

$$\psi_g(\mathbf{x}) = \frac{\kappa_c}{2}\,\|\mathbf{x}\|_2^2 - \frac{\gamma}{2}(\mathbf{x}_1^2 - \mathbf{x}_2^2) + m\sum_{j=1}^{g} \ln \|\mathbf{x} - \xi_j\|_2^2,$$

where $\mathbf{x} = (x_1, x_2)$ and κ_c, γ are physical constants and m represents the mass of the stars. Outside of the random points $\{\xi_i\}_{=1}^{g}$ the potential ψ_g is a C^∞ function. The following random function is known as the delay function for the gravitational lens systems

$$T_{\mathbf{y}}(\mathbf{x}) = \frac{\|\mathbf{x} - \mathbf{y}\|_2^2}{2} - \psi_g(\mathbf{x}).$$

The lensing map is defined as

$$\eta(\mathbf{x}) = \nabla T_{\mathbf{y}}(\mathbf{x}) + \mathbf{y} = \mathbf{x} - \nabla \psi_g(\mathbf{x}).$$

Given the definitions we readily get

$$\eta(\mathbf{x}) = ((1-\kappa_c+\gamma)\mathbf{x}_1, (1-\kappa_c-\gamma)\mathbf{x}_2) - 2m\sum_{j=1}^{g} \frac{\mathbf{x}-\xi_j}{\|\mathbf{x}-\xi_j\|_2^2}.$$

A lensed image is a solution $\mathbf{x}^*$ of the equation $\nabla T_{\mathbf{y}}(\mathbf{x}) = 0$. Thas is

$$\eta(\mathbf{x}^*) = \mathbf{y}.$$

These images correspond to the stationary points of the function $T_{\mathbf{y}}$ and are classified as local maximum, local minimum and saddle point whenever the image is not degenerated. In other case we say that they are degenerated. We are interested in computing the number of non degenerated images having a positive parity N_+ and which are defined as $N_+ = N_{max} + N_{min}$. It is plain to show that in these images the Jacobian of η, that is $\det \nabla\eta(\mathbf{x})$, is always positive.

It results of interest in gravitational studies the computation of the expected number of N_+ generated for a point source $\mathbf{y}$. The number of such images on a set $D \subset \mathbb{R}^2$ is

$$N_+(\mathbf{y}) = \#\{\mathbf{x} \in D : \eta(\mathbf{x}) = \mathbf{y}, \det \nabla\eta(\mathbf{x}) > 0\}.$$

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function with bounded support, then by using the area formula we obtain

$$\int_{\mathbb{R}^2} f(\mathbf{y})\mathbb{E}[N_+(\mathbf{y})]d\mathbf{y} = \int_D \mathbb{E}[f(\eta(\mathbf{x})) \det \nabla\eta(\mathbf{x})\mathbf{1}_{(0,+\infty)}(\det \nabla\eta(\mathbf{x}))]d\mathbf{x}$$

$$= \int_{\mathbb{R}^2} f(\mathbf{y}) \int_D \mathbb{E}[\det \nabla\eta(\mathbf{x})\mathbf{1}_{(0,+\infty)}(\det \nabla\eta(\mathbf{x}))\,|\eta(\mathbf{x}) = \mathbf{y}]p_{\eta(\mathbf{x})}(\mathbf{y})d\mathbf{x}d\mathbf{y}.$$

Although the function η has singularities in the positions of the stars ξ_i these are infinite singularities. That is $\lim_{\mathbf{x}\to\xi_i} \|\eta(\mathbf{x})\|_2 = +\infty$. Hence if we observe only those $\mathbf{y}$ that are in the bounded support of f, we have that the domain of η for each ω is restricted to an open set that does not contain the points ξ_i. This implies that the function η restricted to this set is a C^∞ function. Then the hypothesis for applying the area formula holds.

Moreover, we get for almost surely $\mathbf{y}$

$$\mathbb{E}[N_+(\mathbf{y})] = \int_D \mathbb{E}[\det \nabla\eta(\mathbf{x})\mathbf{1}_{(0,+\infty)}(\det \nabla\eta(\mathbf{x}))\,|\eta(\mathbf{x}) = \mathbf{y}]p_{\eta(\mathbf{x})}(\mathbf{y})d\mathbf{x}.$$

By using the definitions plus some non trivial work, it can be shown that the above formula holds for all $\mathbf{y}$. Furthermore, in [25] the formula is used to get its asymptotic when the number of stars g tends to infinite. An interesting but still open problem is to get the same asymptotic for the variance of $N_+(\mathbf{y})$.

4.2.4 Kostlan-Shub-Smale systems

Consider a rectangular system $\mathbf{P} = 0$ of j polynomial equations in $d \geqslant j$ variables. We assume that the equations have the same degree $n > 1$. Let $\mathbf{P} = (P_1, \dots, P_j)$, we can write each polynomial P_ℓ in the form

$$P_\ell(\mathbf{t}) = \sum_{|\mathbf{z}| \leq n} a_{\mathbf{z}}^{(\ell)} \mathbf{t}^{\mathbf{z}},$$

where

1. $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{N}^d$ and $|\mathbf{z}| = \sum_{k=1}^d z_k$;
2. $a_{\mathbf{z}}^{(\ell)} = a_{z_1 \dots z_d}^{(\ell)} \in \mathbb{R}$, $\ell = 1, \dots, j$, $|\mathbf{z}| \leq n$;
3. $\mathbf{t} = (t_1, \dots, t_d)$ and $t^{\mathbf{z}} = \prod_{k=1}^d t_k^{z_k}$.

We say that $\mathbf{P}$ has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{\mathbf{z}}^{(\ell)}$ are independent centered normally distributed random variables with variances

$$\text{Var}\left(a_{\mathbf{z}}^{(\ell)}\right) = \binom{n}{\mathbf{z}} = \frac{n!}{z_1! \dots z_d!(n - |\mathbf{z}|)!}.$$

We are interested in the set of zeros of $\mathbf{P}$. We denote by $N_n^{\mathbf{P}}$ its cardinal if $d = j$ or if $d > j$ we denote this set as $\mathcal{C}_{\mathbf{P}}(\mathbf{0})$ and its volume as $\mathcal{L}(\mathcal{C}_{\mathbf{P}}(\mathbf{0}))$. Shub and Smale [22] proved that if $d = j$ then $\mathbb{E}(N_n^{\mathbf{P}}) = n^{d/2}$. In chapter 12 of Azaïs & Wschebor book [6] this result was obtained by using the Kac-Rice formula. Letendre in [21] has tackled the case $d > j$ obtaining the following result firstly showed by Kostlan in [19]. It holds

$$\mathbb{E}[\mathcal{L}(\mathcal{C}_{\mathbf{P}}(\mathbf{0}))] = n^{\frac{j}{2}} \frac{\pi^{\frac{(d-j+1)}{2}}}{\Gamma[\frac{(d-j+1)}{2}]}.$$

Below following Letendre's method and some simplifications we will obtain this result by using the Kac-Rice formula.

It is customary and convenient to homogenize the polynomials. That is, to add an auxiliary variable t_0 in order that all the monomials have the same degree n. More precisely, we multiply the monomial in P_ℓ corresponding to the index $\mathbf{z}$ by $t_0^{n-|\mathbf{z}|}$. Let

$$\mathbf{X} = (X_1, \dots, X_j),$$

denote the resulting vector of j homogeneous polynomials in $d+1$ real variables with common degree $n > 1$. We have,

$$X_\ell(\mathbf{t}) = \sum_{|\mathbf{z}|=n} a_{\mathbf{z}}^{(\ell)} \mathbf{t}^{\mathbf{z}}, \quad \ell = 1, \dots, j,$$

where this time $\mathbf{z} = (z_0, \dots, z_d) \in \mathbb{N}^{d+1}$;

$$|\mathbf{z}| = \sum_{k=0}^{d} z_k; a_j^{(\ell)} = a_{z_0 \dots z_d}^{(\ell)} \in \mathbb{R};$$

$\mathbf{t} = (t_0, \dots, t_d) \in \mathbb{R}^{d+1}$ and $\mathbf{t}^{\mathbf{z}} = \prod_{k=0}^{d} t_k^{z_k}$.

Each X_ℓ is homogeneous, and the zero set of $\mathbf{X}$ is the intersection of the zero set of each X_ℓ. Then the set $\mathcal{C}_{\mathbf{X}}(\mathbf{0})$ is a subset of the real projective space $\mathbb{RP}^{d-j}$.

From now on we work with the homogenized version $\mathbf{X}$. Standard multinomial formula shows that for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}^{d+1}$ we have

$$r_d(\mathbf{s}, \mathbf{t}) := \Gamma_n(\langle \mathbf{s}, \mathbf{t} \rangle) := \mathbb{E}\left(X_\ell(\mathbf{s}) X_\ell(\mathbf{t})\right) = \langle \mathbf{s}, \mathbf{t} \rangle^n,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^{d+1}$. As a consequence, we see that the distribution of the system $\mathbf{X}$ is invariant under the action of the orthogonal group in $\mathbb{R}^{d+1}$. We see also that the distribution depends of course of n and this will be omitted for $\mathbf{X}$ for the ease of notation.

In the sequel we need to consider the derivative of X_ℓ, $\ell = 1, \dots, j$. Since the parameter space is the sphere $\mathbb{S}^d$, the derivative is taken in the sense of the sphere, that is, the spherical derivative $X'_\ell(\mathbf{t})$ of $X_\ell(\mathbf{t})$ is the orthogonal projection of the free gradient on the tangent space $\mathbf{t}^\perp$ of $\mathbb{S}^d$ at $\mathbf{t}$. The k-th component of $X'_\ell(\mathbf{t})$ at a given basis of the tangent space is denoted by $X'_{\ell k}(\mathbf{t})$.

We are going to use the Rice formula with a slight modification to make it valid on $\mathbb{S}^d$. As the process $\mathbf{X}$ satisfies the hypotheses of Remark 3.3.1 following Theorem 3.3.1 we get

$$\mathbb{E}[\mathcal{L}(\mathcal{C}_{\mathbf{X}}(\mathbf{0}))]$$
$$= \int_{\mathbb{S}^d} \mathbb{E}[(\det(\nabla\mathbf{X}(\mathbf{t}))(\nabla\mathbf{X}(\mathbf{t}))^T)^{\frac{1}{2}} \,|\, |\mathbf{X}(\mathbf{t}) = \mathbf{0}] p_{\mathbf{X}(\mathbf{t})}(\mathbf{0}) \gamma_d(d\mathbf{t}),$$

where γ_d stands for the d-dimensional geometric measure on $\mathbb{S}^d$. Since $\mathbf{E}[\mathbf{X}(\mathbf{t})^2] = 1$, $\mathbf{X}(\mathbf{t})$ and $\nabla\mathbf{X}(\mathbf{t})$ are independent, allowing to erase the conditioning into the expectation. Furthermore as $p_{\mathbf{X}(\mathbf{t})}(\mathbf{0}) = \frac{1}{(2\pi)^{\frac{j}{2}}}$, if $\nabla\mathbf{X}(\mathbf{e}_0)$ stands for the matrix with generic element $X'_{\ell k}(\mathbf{0})$, we finally get

$$\begin{aligned}
\mathbb{E}[\mathcal{L}(\mathcal{C}_{\mathbf{X}}(\mathbf{0}))] &= \frac{\sigma_d(\mathbb{S}^d)}{(2\pi)^{\frac{j}{2}}} \mathbb{E}[(\det(\nabla\mathbf{X}(\mathbf{e}_0))(\nabla\mathbf{X}(\mathbf{e}_0))^T)^{\frac{1}{2}}] \\
&= n^{j/2} \frac{\sigma_d(\mathbb{S}^d)}{(2\pi)^{\frac{j}{2}}} \mathbb{E}[(\det(\nabla\mathbf{Z}(\mathbf{e}_0))(\nabla\mathbf{Z}(\mathbf{e}_0))^T)^{\frac{1}{2}}],
\end{aligned}$$

where $\nabla\mathbf{Z}(\mathbf{e}_0)$ is $N(0, I_j)$. But a Gaussian computation (cf. [21]) gives

$$E[(\det(\nabla\mathbf{Z}(\mathbf{e}_0))(\nabla\mathbf{Z}(\mathbf{e}_0))^T)^{\frac{1}{2}}] = (2\pi)^{\frac{j}{2}} \frac{\sigma_{d-j}(\mathbb{S}^{d-j})}{\sigma_d(\mathbb{S}^d)}.$$

Yielding

$$\mathbb{E}[\mathcal{L}(\mathcal{C}_{\mathbf{X}}(\mathbf{0}))] = n^{j/2} \sigma_{d-j}(\mathbb{S}^{d-j}) = n^{j/2} \frac{\pi^{\frac{d-j+1}{2}}}{\Gamma[\frac{d-j+1}{2}]}.$$

The variance of this random variable has been also computed by Letendre and it will be interesting to prove a CLT.

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